Background
Solutions and Initial Value Problems

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In single variable calculus, we use techniques of integration that allow us to determine a solution to

$$y'(x) = f(x)$$

with $y(a) = c$ by evaluating the integral

$$y(x) = c + \int_a^x f(t)dt.$$

For differential equations, the term **ordinary** indicates that we will be consider y along with its derivatives $y', y'', \ldots.$ as functions of a single variable x. We can write this differential equation as

$$F(x, y(x), y'(x), y''(x), \ldots, y^{(n)}(x)) = 0 \tag{1}$$

or if we can isolate the higherst order derivative, as

$$y^{(n)}(x) = f(x, y(x), y'(x), y''(x), \ldots, y^{(n-1)}(x)).$$

The highest derivative in this expression, n, is called the **order** of the differential equation.

An ordinary differential equation is called **linear** if $F(x, y(x), y'(x), y''(x), \ldots, y^{(n)}(x))$ can be written in the form

$$a_n(x)y^{(n)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x).$$

If a differential equation is not linear, it is called **non-linear**.

Ordinary differential equations are used to model and analyze any situation in which the change in some function depends on the value of the function. These models can be seen in mechanical motion, dynamical properties of electrical circuits (physics), the motion of celestial bodies (astronomy), chemical reactions (chemistry), changes in land forms (geology), heat transfer and thermodynamics (mechanical engineering), properties of materials (material science and optics), population models (ecology and epidemiology), and the flow of capital and goods (economics)

Example 1. *A typical model for the dynamics of a populations is a* **logistic growth model**

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right).$$

So if $P << K$, $P/K \approx 0$, and

$$\frac{dP}{dt} \approx rP. \tag{2}$$

Question: *If the population $P = 0$ then what is the rate of change of the population size? Why does this makes sense?*

Question: *If the population $P = K$ then what is the rate of change of the population size? What does this mean?*

We can **separate variables** *and integrate to find a soluton*

$$\begin{aligned}
\frac{1}{P(1-P/K)}\frac{dP}{dt} &= r \\
\int \frac{1}{P(1-P/K)}\frac{dP}{dt}dt &= \int r dt \\
\int \frac{1}{P(1-P/K)}dP &= rt + c \quad \textit{(w-substitution)} \\
\int \left(\frac{1}{P} + \frac{1}{(K-P)}\right) dP &= rt + c \quad \textit{(Check the partial fraction decomposition.)} \\
\ln P - \ln(K-P) &= rt + c \\
\ln\left(\frac{P}{K-P}\right) &= rt + c \\
\frac{P}{K-P} &= Ae^{rt} \quad \textit{where } A = e^c \\
P(t) &= \frac{KAe^{rt}}{1+Ae^{rt}} = \frac{KA}{A+e^{-rt}} \quad \textit{(Check the algebra.)}
\end{aligned}$$

At time $t = 0$,

$$P_0 = P(0) = \frac{KA}{1+A}$$

Exercise 2. *Find a solution to*

$$\frac{dP}{dt} = rP.$$

Interpret this answer in terms of (2).

Exercise 3. *Find A in terms of P_0 and K.*

Exercise 4. *Find $\lim_{t\to\infty} P(t)$. What does this mean in practical terms?*

We can check if a function ϕ is an **explicit solution** to a differential equation on an interval I by substituting into the equation and verifying that the equation holds for all $x \in I$

Exercise 5. *Verify that $\phi(x) = x^{-3/2}$ is a solution to $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$.*

Exercise 6. *Verify that $\phi(x) = A\cos(2x) + B\sin(2x)$ is a solution to $y'' = -4y$ for $x > 0$.*

Sometimes the the relationship between x and y is only known implicitly, i.e.,

$$G(x, y) = 0$$

for some function G. In this case, G is called an **implicit solution** to a differential equation is one that defines one or more explicit solutions.

Exercise 7. *$y^2 = x^2 + 4$ is an implicit solution to the differential equation*

$$y' = \frac{x}{y}, \quad y(0) = 2 \tag{3}$$

Definition 8. *An* **initial value problem** *for the n-th order differential equation (1) poses the question: Find a solution to the differential equation on an interval I so that*

$$y(x_0) = y_0,\ y'(x_0) = y_1 \ldots, y^{(n-1)} = y_{n-1} \tag{4}$$

for constants $y_0, y_1, \ldots y_{n-1}$ and for $x \in I$.

Exercise 9. *For the implicit solution above,*

$$y(x) = \pm\sqrt{x^2 + 4}.$$

Which choice is the solution to the initial value problem (3)?

Exercise 10. *Show that*

$$y(x) = 2e^{-3x} + 3e^{2x}$$

is a solution to the initial value problem

$$y'' + y' - 6y = 0,$$

with $y(0) = 5$ and $y'(0) = 0$.

The two fundamental questions we have for the initial value problem (4) is

- Under what conditions does an explicit solution exist?
- Under what conditions can we be sure that there is a unique solution?

For first order equations we have the following theorem.

Theorem 11. *For the initial value problem*

$$y' = f(x, y), \quad y(x_0) = y_0,$$

Assume that both f and the partial derivative $\partial f/\partial y$ are continuous is some rectangle $R = (a, b) \times (c, d)$ that contains the initial value (x_0, y_0), then, for some $\delta > 0$, there exists a unique solution for $x \in (x_0 - \delta, x_0 + \delta)$.

Exercise 12. *Does the theorem above apply to the initial value problem*

1. $y' = 2x^2 + \cos(y), \quad y(0) = 0.$
2. $y' = 2\sqrt{y}, \quad y(2) = 0.$