Conditional Probability and Independence

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1 Restricting the Sample Space - Conditional Probability

How do we modify the probability of an event in light of the fact that something is known?

If the top card is 3♠, what is the probability that the second card is a three? a ♠? a king?

All of your answers have 51 in the denominator. You have mentally restricted the sample space from S with 52 outcomes to $B = \{\text{all cards but } 3\spadesuit\}$ with 51 outcomes. We call the answer the **conditional probability**.

For equally likely outcomes, we have a formula.

$$\begin{aligned} P(A|B) &= \text{the proportion of outcomes in } A \text{ that are also in } B \\ &= \frac{\#(A \cap B)}{\#(B)} = \frac{\#(A \cap B)/\#(S)}{\#(B)/\#(S)} \end{aligned}$$

This allows us to answer the questions above.

$$P\{\text{second card is a three}|\text{first card is } 3\spadesuit\} = \frac{3}{51} = \frac{1}{17}$$

$$P\{\text{second card is a } \spadesuit|\text{first card is } 3\spadesuit\} = \frac{\quad}{51}$$

$$P\{\text{second card is a king}|\text{first card is } 3\spadesuit\} = \frac{\quad}{51}$$

The last identity for $P(A|B)$ with equally likely outcomes can be interpreted as the ratio of probabilities:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

We thus take this version of the identity as the general definition of conditional expectation for any pair of events A and B as long as $P(B) > 0$

Exercise 1. *If $A \subset B$, then $P(A|B) = P(A)/P(B)$. If $B \subset A$, then $P(A|B) = 1$.*

Example 2. *For two rolls of a die, we have 36 possible outcomes.*

$$P\{\text{sum is } k\} = \frac{6 - |k-7|}{36}.$$

Now suppose that we are to roll two dice until on of two numbers k_1 and k_2 is observed for the sum. Our sample space S is the set of all outcomes that sum to k_1 or k_2. Thus,

$$\begin{aligned} P\{\text{sum is } k_1|\text{sum is } k_1 \text{or } k_2\} &= \frac{P\{\text{sum is } k_1\}}{P\{\text{sum is } k_1 \text{ or } k_2\}} = \frac{(6-|k_1-7|)/36}{(6-|k_1-7|+6-|k_2-7|)/36} \\ &= \frac{6-|k_1-7|}{12-|k_1-7|-|k_2-7|} \end{aligned}$$

2 The Multiplication Principle

Multiply both sides of the defining formula for conditional probability by $P(B)$ to obtain **the multiplication principle**

$$P(A \cap B) = P(A|B)P(B)$$

Now, we can complete an earlier problem

$$\begin{aligned} P\{\text{ace on first two cards}\} &= P\{\text{ace on second card}|\text{ace on first card}\}P\{\text{ace on first card}\} \\ &= \frac{1}{17} \times \frac{1}{13}. \end{aligned}$$

We can continuue this process to obtain a **chain rule**

$$P(A \cap B \cap C) = P(A|B \cap C)P(B \cap C) = P(A|B \cap C)P(B|C)P(C).$$

Tbus,

$$\begin{aligned} &P\{\text{ace on first three cards}\} \\ &= P\{\text{ace on third card}|\text{ace on first and second card}\}P\{\text{ace on second card}|\text{ace on first card}\}P\{\text{ace on first card}\} \\ &= \frac{1}{26} \times \frac{1}{17} \times \frac{1}{13}. \end{aligned}$$

Extending this 4 events, we have:

Example 3. *In a urn with b blue balls and g green balls, the probability of green, blue, green, blue is*

$$\frac{g}{b+g} \cdot \frac{b}{b+g-1} \cdot \frac{g-1}{b+g-2} \cdot \frac{b-1}{b+g-3} = \frac{(g)_2(b)_2}{(b+g)_4}$$

Exercise 4. *What is the probability of 2 blue and 2 green in four draws from the urn above?*

3 Law of Total Probability

Definition 5. *A* **partition** *of the sample space S if a finite collection of pairwise mutually exclusive events $\{C_1, C_2, \ldots, C_n\}$ whose union is S.*

Thus, every point $s \in S$ belong to *exactly* one of the C_i.

Theorem 6 (Law of total probability)**.** *Let P be a probability on S. and let $\{C_1, C_2, \ldots, C_n\}$ be a partition of S chosen so that $P(C_i) > 0$ for all i. Then, for any event $A \subset S$*

$$P(A) = \sum_{i=1}^{n} P(A|C_i)P(C_i).$$

Proof. Because $\{C_1, C_2, \ldots, C_n\}$ is a partition, $\{A \cap C_1, A \cap C_2, \ldots, A \cap C_n\}$ are pairwise mutually exclusive events. By the distributive property of sets, their union is A. Thus,

$$P(A) = \sum_{i=1}^{n} P(A \cap C_i).$$

Finish by using the identity

$$P(A \cap C_i) = P(A|C_i)P(C_i), \quad i = 1, 2, \ldots, n.$$

□

Example 7 (craps)**.** *To play craps, first role two dice.*

- *If the sum is 7 or 11, then the player wins immediately.*
- *If the sum is 2, 3, or 12, then the player loses immediately.*
- *If the sum is 4, 5, 6, 8, 9, or 10, if if the player rolls this number a second time before rolling a 7, then the player wins*

So

$$P\{\text{winning immediately}\} = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}.$$

For $k = 4, 5, 6, 8, 9$, or 10,

$$\begin{aligned} P\{\text{winning with } k\} &= P\{\text{rolling } k \text{ before } 7|\text{first roll is } k\} \cdot P\{\text{first roll is } k\} \\ &= \frac{6-|k-7|}{12-|k-7|} \cdot \frac{6-|k-7|}{36} = \frac{(6-|k-7|)^2}{36(12-|k-7|)} \end{aligned}$$

This yields the table

first roll	4	5	6	8	9	10
probability of winning	$\frac{9}{36\cdot 9}$	$\frac{16}{36\cdot 10}$	$\frac{25}{36\cdot 11}$	$\frac{25}{36\cdot 11}$	$\frac{16}{36\cdot 10}$	$\frac{9}{36\cdot 9}$

Thus,

$$P\{\text{winning after the first toss}\} = \frac{2}{36}\left(1 + \frac{8}{5} + \frac{25}{11}\right) = .\frac{1}{18}\left(\frac{55+88+125}{55}\right) = .\frac{268}{18\cdot 55} = .\frac{134}{9\cdot 55}$$

and

$$P\{\text{winning}\} = \frac{2}{9} + \frac{134}{9\cdot 55} = \frac{110+134}{9\cdot 55} = \frac{244}{495} = 0.493.$$

4 Bayes formula

Theorem 8 (Bayes formula)**.** *Let P be a probability on S. and let $\{C_1, C_2, \ldots, C_n\}$ be a partition of Schosen so that $P(C_i) > 0$ for all i. Then, for any event $A \subset S$ and any j*

$$P(C_j|A) = \frac{P(A|C_j)P(C_j)}{\sum_{i=1}^n P(A|C_i)P(C_i)}.$$

Proof. By the law of total probability, the dominator is equal to $P(A)$. By the definition of conditional probability, the numerator is equal to $P(C_j \cap A)$. Now, make these two substitutions and use one more time the definition of conditional probability. □

Example 9. *The Public Health Department gives us the following information.*

- *A test for the disease yields a positive result 90% of the time when the disease is present.*
- *A test for the disease yields a positive result 10% of the time when the disease is not present.*
- *One person in 10,000 has the disease.*

Let

- *C be the event of having the disease.*
- *A be the event of testing positive for the disease.*

Then

$$P(A|C) = 0.90, \quad P(A|C^c) = 0.10, \quad P(C) = 0.0001$$

and

$$P(A) = P(A|C)P(C) + P(A|C^c)P(C^c) = 0.90 \cdot 0.0001 + 0.10 \cdot 0.9999 = 0.00009 + 0.09999 = 0.10008.$$

Thus, the probability of having the disease given that the test was positive is

$$P(C|A) = \frac{P(A|C)P(C)}{P(A)} = \frac{0.00009}{0.10008} = 0.00089.$$

5 Independence

An event A is **independent of** B if

$$P(A) = P(A|B).$$

In other words, the knowledge of B does not influence $P(A)$. Multiply both sides of the formula in the definition of independence by $P(B)$ and use the multiplication rule to obtain

$$P(A)P(B) = P(A|B)P(B) = P(A \cap B)$$

This last formula is the usual definition of independence and is symmetric in the events A and B. We can also use this to extend the definition to n independent events

The events $A_1, \cdots, A_n$ are **independent** if

$$P(A_1 \cap \cdots \cap A_n) = P(A_1) \times \cdots \times P(A_n).$$

Exercise 10. *If A and B are independent, then show that A^c and B, A and B^c, A^c and B^c are also independent.*

Roll two dice.

$$\begin{aligned} \frac{1}{36} &= P\{a \text{ on the first die}, b \text{ on the second die}\} \\ &= \frac{1}{6} \times \frac{1}{6} = P\{a \text{ on the first die}\}P\{b \text{ on the second die}\} \end{aligned}$$

and, thus, the outcomes on two rolls of the dice are independent.