The Approximation Method of Euler

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In many circumstances, we will not be able to use the techniques of calculus to determine the solution for the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0.$$

We consequently rely on numerical techniques to approximate a solution. The first is **Euler's method**. This method relies on the tangent line or the first order Tyler's approximation an differential function.

Begin by choosing a (small) positive number h, Then

$$y(x + h) \approx y(x) + hy'(x) = y(x) + hf(x, y).$$

This tangent line approximation is used iteratively beginning with the point (x_0, y_0)

$$\begin{array}{rclrcl} x_1 &=& x_0 + h & y_1 &=& y_0 + hf(x_0, y_0) \\ x_2 &=& x_1 + h = x_0 + 2h & y_2 &=& y_1 + hf(x_1, y_1) \\ & & \vdots & & & \\ x_{n+1} &=& x_n + h = x_0 + nh & y_{n+1} &=& y_n + hf(x_n, y_n) \end{array}$$

Continue the iteration until some final value for x_f is obtained. In this case,

$$n = \frac{x_f - x_0}{h}.$$

Let's implement the Eurer method for the differential equation for the differential equation,

$$y' = \frac{x}{y}, \quad y(0) = 2 \tag{1}$$

We know that the solution is $y(x) = \sqrt{x^2 + 4}$.

Take $h = 0.1$ and $x_f = 1$. So, $y(1) = \sqrt{5}$. Here is the implementation in R

```
> f<-function(x,y) x/y
> x0<-0;xf<-1;y0<-2;h<-0.1
> n<-ceiling((xf-x0)/h);x<-numeric(n);y<-numeric(n)
> x[1]<=<-h;y[1]<-y0+h*f(x0,y0)
> for (i in 2:n){x[i]<-x0+i*h;y[i]<-y[i-1]+h*f(x[i-1],y[i-1])}
```

We print the output in a `data.frame`

```
> data.frame(x,y,sqrt(x^2+4))
     x        y sqrt.x.2...4.
1  0.1 2.000000      2.002498
2  0.2 2.005000      2.009975
3  0.3 2.014975      2.022375
4  0.4 2.029864      2.039608
5  0.5 2.049569      2.061553
6  0.6 2.073965      2.088061
7  0.7 2.102895      2.118962
8  0.8 2.136182      2.154066
9  0.9 2.173632      2.193171
10 1.0 2.215038      2.236068
```

The **relative error** is

$$\left|\frac{\text{exact} - \text{approximation}}{\text{exact}}\right|.$$

For this case, we have the table for the relative error.

h	$y(1)$	relative error
0.1	2.2105	1.14×10^{-2}
0.01	2.2339	9.49×10^{-4}
0.001	2.2359	9.29×10^{-5}

Exercise 1. *Check that* $y(x) = 2e^x - x - 1$ *is an explicit solution to*

$$y' = x + y, \quad y(0) = 2$$

. *Compare this to Euler's method with* $h = 0.1$ *and* $x = 0.1, 0.2, \ldots, 1.0$.