Separable Equations
Linear Equations

June 10, 2016

Again we return to the initial value problem.

$$y = f(x, y), \quad y(x_0) = y_0 \tag{1}$$

1 Separable Equations

We will now look at a variety of forms of the function f that will allow us to find an analytic solution to (1). The first is a case of **separable equations**. In this case, we will be able to reduce the problem to that of two integrals - one for x and one for y - followed by some algebraic manipulations to invert a function and isolate y

If f factorizes, that is, $f(x, y) = p(x)q(y)$, then (1). becomes

$$\begin{aligned}
\frac{dy}{dx} &= f(x, y) = p(x)q(y) \\
\frac{1}{q(y)}\frac{dy}{dx} &= p(x) \\
\int \frac{1}{q(y)}\frac{dy}{dx}dx &= \int p(x)dx \\
\int \frac{1}{q(y)}dy &= \int p(x)dx \\
Q(y) &= P(x) + c
\end{aligned}$$

Here P is an antiderivative for p. Q is an antiderivative for $1/q$. This gives an implicit solution. To find an explicit solution, we solve for y.

$$y = Q^{-1}(P(x) + c).$$

Finally c is chosen so that the initial condition is satisfied. Thus,

$$y_0 = Q^{-1}(P(x_0) + c).$$

These abstract series of equations will become more understandable after we complete several examples.

Example 1. *To find the solutions to*

$$y' = \frac{y^2-1}{x}, \quad y(1) = 2$$

we follow the steps above.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y^2-1}{x} \\
\frac{1}{y^2-1}\frac{dy}{dx} &= \frac{1}{x} \\
\int \frac{1}{y^2-1}\frac{dy}{dx}dx &= \int \frac{1}{x}dx \\
\int \frac{1}{y^2-1}dy &= \int \frac{1}{x}dx \\
\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| &= \ln|x| + c \quad \textit{Check the integral} \\
\sqrt{\left|\frac{y-1}{y+1}\right|} &= e^c|x|
\end{aligned}$$

Before we solve for y, we will use the initial conditions to help us with the absolute values.

$$\sqrt{\left|\frac{2-1}{2+1}\right|} = e^c|1|, \quad \sqrt{\frac{1}{3}} = e^c$$

In both of these case the number inside the absolute value is positive and so we can drop $|\cdot|$ and

$$\begin{aligned}
\sqrt{\frac{y-1}{y+1}} &= \sqrt{\frac{1}{3}}x \\
\frac{y-1}{y+1} &= \frac{1}{3}x^2 \\
y &= \frac{3+x^2}{3-x^2} \quad \textit{Check the algebra.}
\end{aligned}$$

Example 2. *For*

$$y' = \frac{x}{2+\cos y}, \quad y(2) = 0$$

Again we have the separation of variables.

$$\begin{aligned}
(2+\cos y)\frac{dy}{dx} &= x \\
\int (2+\cos y)\frac{dy}{dx}dx &= \int xdx \\
\int (2+\cos y)dy &= \int xdx \\
2y - \sin y &= \frac{1}{2}x^2 + c
\end{aligned}$$

For the initial consition,

$$2 \cdot 0 - \sin 0 = \frac{1}{2}2^2 + c, \quad 0 = 2 + c, \quad c = -2$$

This gives the implicit solution

$$2y - \sin y = \frac{1}{2}x^2 - 2.$$

An explicit solution would require solving this equation for y.

Exercise 3. *Use the method above for separable equations to solve*

- $\frac{dy}{dx} = \frac{(1-x^2)y^3}{x^2}$, $y(1) = 1$.
- $\frac{dy}{dx} = \frac{x(\exp(x^2)+1)}{6y^2}$, $y(0) = 1$.

2 Motion of a Falling Body

Example 4. *Newton's law, force is the mass times the acceleration, can be written*

$$F = m\frac{dv}{dt}$$

by noting that the acceleration is the derivative of the velocity.

We will consider two forces - gravity, which for an object is a force downward proportional to the mass and the acceleration of gravity, and air resistance with is proportional to the velocity. The constant of proportionality is negative to indicate that the force acts in the opposite direction to that of the force of gravity. Thus,

$$\begin{aligned} m\frac{dv}{dt} &= mg - bv \\ \frac{dv}{dt} &= g - \frac{b}{m}v \end{aligned}$$

We can apply the separation of variables technique to this case

$$\begin{aligned} \frac{1}{g - bv/m}\frac{dv}{dt} &= 1 \\ \int \frac{1}{g - bv/m}\frac{dv}{dt}dt &= \int 1dt \\ \int \frac{1}{g - bv/m}dv &= \int 1dt \\ -\frac{m}{b}\ln\left|g - \frac{b}{m}v\right| &= t + c \quad \textit{Check the integral} \\ \ln\left|g - \frac{b}{m}v\right| &= -\frac{m}{b}(t + c) \\ \left|g - \frac{b}{m}v\right| &= \exp(-\frac{m}{b}(t + c)) \\ \left|g - \frac{b}{m}v\right| &= A\exp(-\frac{m}{b}t), \quad A = \exp(-\frac{m}{b}c), \ \textit{a constant} \end{aligned}$$

This **general solution** *awaits an initial condition to determine the value of A. We can absorb the sign of the absolute into the value for A and find the initial condition.*

$$g - \frac{b}{m} v_0 = A$$

$$\begin{aligned}
g - \frac{b}{m} v &= (g - \frac{b}{m} v_0) \exp(-\frac{m}{b} t) \\
g - (g - \frac{b}{m} v_0) \exp(-\frac{m}{b} t) &= \frac{b}{m} v \\
\frac{mg}{b} - \frac{m}{b}(g - \frac{b}{m} v_0) \exp(-\frac{m}{b} t) &= v \\
\frac{mg}{b} + \left(v_0 - \frac{m}{b} g\right) \exp(-\frac{m}{b} t) &= v
\end{aligned}$$

Notice that

$$\lim_{t \to \infty} v(t) = \frac{mg}{b},$$

known as the terminal velocity. If

- *If $v_0 < mg/b$, then the object's velocity increases to the terminal velocity.*
- *If $v_0 > mg/b$, then the object's velocity decreases to the terminal velocity.*
- *The terminal velocity increases with mass and in inversely proportional to distance.*

3 Linear Equations

We now look at first order linear differential equation. These equations take the form

$$a_1(x) \frac{dy}{dx} + a_0(x) y = b(x). \tag{2}$$

Let's begin with an example that we can later generalize.
Consider

$$x^2 \frac{dy}{dx} + 2xy = \sin x$$

Notice that the left side is the derivative of a product.

$$\frac{d}{dx}(x^2 y) = \sin x$$

Now integrate both sides

$$\begin{aligned}
\int \frac{d}{dx}(x^2 y) dx &= \int \sin x dx \\
x^2 y &= -\cos x + c \\
y &= \frac{-\cos x + c}{x^2}
\end{aligned}$$

This is a general solution. We can then use initial conditions to determine c, the constant of integration. The key feature for this procedure is for a_0 to be the derivative of a_1. So

$$\begin{aligned}
a_1(x)\frac{dy}{dx} + a_0(x)y &= b(x) \\
a_1(x)\frac{dy}{dx} + a_1'(x)y &= b(x) \\
\frac{d}{dx}(a_1(x)y) &= b(x) \\
\int \frac{d}{dx}(a_1(x)y)dx &= \int b(x)dx \\
a_1(x)y &= B(x) + c \quad \text{where } B(x) \text{ is an antiderivative for } b(x) \\
y &= \frac{B(x)+c}{a_1(x)}
\end{aligned}$$

We shall learn that this procedure can alway be achieved with one extra step. To show this, we first give the (3) by dividing by $a_1(x)$ so that the coefficient in front of the derivative term is 1.

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{3}$$

So $P(x) = a_0(x)/a_1(x)$ and $Q(x) = b(x)/a_1(x)$.

For the simplest case

$$\frac{dy}{dx} + y = \sin x,$$

our goal is to find a function $\mu(x)$ (called the **integrating factor**) so that

$$\mu(x)\frac{dy}{dx} + \mu(x)y = \mu(x)\sin x,$$

and $a_0(x) = \mu(x)$ is the derivative of $a_1(x) = \mu(x)$

In other words,

$$\mu'(x) = \mu(x).$$

A solution to this is $\mu(x) = \exp(x)$. Returning to our equation

$$\begin{aligned}
\exp(x)\frac{dy}{dx} + \exp(x)y &= \exp(x)\sin(x) \\
\frac{d}{dx}(\exp(x)y) &= \exp(x)\sin(x) \\
\int \frac{d}{dx}(\exp(x)y)dx &= \int \exp(x)\sin(x)dx \\
\exp(x)y &= \frac{1}{2}\exp(x)(\sin x - \cos x) + c \quad \text{(Check this.)} \\
y &= \frac{1}{2}(\sin x - \cos x) + c\exp(-x)
\end{aligned}$$

Exercise 5. *Verify that the solution above is an explicit solution to* $\frac{dy}{dx} + y = \sin x$.

Now, let's se if we can do this in general for (3).

$$\begin{aligned}\frac{dy}{dx}+P(x)y &= Q(x)\\ \mu(x)\frac{dy}{dx}+\mu(x)P(x)y &= \mu(x)Q(x)\end{aligned}$$

Now the condition that $a_0'(x) = a_1(x)$ yields

$$\begin{aligned}\mu'(x) &= \mu(x)P(x)\\ \frac{\mu'(x)}{\mu(x)} &= P(x)\\ \int\frac{\mu'(x)}{\mu(x)}dx &= \int P(x)dx\\ \ln\mu(x) &= \int P(x)dx\\ \mu(x) &= \exp\left(\int P(x)dx\right)\end{aligned}$$

Then,

$$\begin{aligned}\mu(x)\frac{dy}{dx}+\mu(x)P(x)y &= \mu(x)Q(x)\\ \exp\left(\int P(x)dx\right)\frac{dy}{dx}+\exp\left(\int P(x)dx\right)P(x)y &= \exp\left(\int P(x)dx\right)Q(x)\\ \frac{d}{dx}\left(\exp\left(\int P(x)dx\right)y\right) &= \exp\left(\int P(x)dx\right)Q(x)\\ \int\frac{d}{dx}\left(\exp\left(\int P(x)dx\right)y\right)dx &= \int\exp\left(\int P(x)dx\right)Q(x)dx+c\\ \exp\left(\int P(x)dx\right)y &= \int\exp\left(\int P(x)dx\right)Q(x)dx+c\\ y &= \exp\left(-\int P(x)dx\right)\left(\int\exp\left(\int P(x)dx\right)Q(x)dx+c\right)\end{aligned}$$

In the examples, rather than use the abstract equation directly, we will work with the method, Find μ and then place it in the the differential equation to solve.

Example 6. *Returntng to the equation for the falling body,*

$$m\frac{dv}{dt} = mg - bv,$$

we place it in the form found in (3).

$$\frac{dv}{dt}+\frac{b}{m}v = g.$$

The integrating factor

$$\mu(t) = \exp\left(\int \frac{b}{m} dt\right) = \exp\left(\frac{b}{m} t\right)$$

This give the equation

$$\begin{aligned}
\exp\left(\frac{b}{m}t\right)\frac{dv}{dt} + \exp\left(\frac{b}{m}t\right)\frac{b}{m}v &= \exp\left(\frac{b}{m}t\right)g. \\
\frac{d}{dt}\left(\exp\left(\frac{b}{m}t\right)v\right) &= \exp\left(\frac{b}{m}t\right)g \\
\exp\left(\frac{b}{m}t\right)v &= \int \exp\left(\frac{b}{m}t\right)g = \exp\left(\frac{b}{m}t\right)\frac{gm}{b} + c \\
v &= \exp\left(-\frac{b}{m}t\right)\left(\exp\left(\frac{b}{m}t\right)\frac{gm}{b} + c\right) \\
v &= \exp\left(-\frac{b}{m}t\right)\left(\exp\left(\frac{b}{m}t\right)\frac{gm}{b} + c\right) \\
v &= \left(\frac{gm}{b} + c\exp\left(-\frac{b}{m}t\right)\right)
\end{aligned}$$

Example 7. *For the differential equation*

$$\cos(x)\frac{dy}{dx} + \sin(x)y = \cos^2(x),$$

we write it in the form (3).

$$\frac{dy}{dx} + \tan(x)y = \cos(x)$$

The integrating factor

$$\mu(x) = \exp\left(\int \tan(x)dx\right) = \exp(\ln|\sec(x)|) = |\sec(x)|.$$

For $\sec(x) > 0$*, we have*

$$\begin{aligned}
\sec(x)\frac{dy}{dx} + \sec(x)\tan(x)y &= 1 \\
\frac{d}{dx}(\sec(x)y) &= 1 \\
\sec(x)y &= x + c \\
y &= (x + c)\cos(x)
\end{aligned}$$