Exact Equations

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1 Connection to Conservative Vector Fields

In vector calculus, we studied two dimensional vector fields

$$F(x,y) = (M(x,y), N(x,y)). \tag{1}$$

For every point (x,y) in the plane, we have a vector $F(x,y)$ with x component $M(x,y)$ and y component $N(x,y)$.

If the vector field is called **conservative**, the F if the line integral is path independent between two points, i.e. the value of integral from a point A to a point B does not depend on the choice of path. In these cases, F can be realized as the gradient of a scalar function f, called the **potential**. Thus,

$$F(x,y) = \nabla f(x,y) = \left(\frac{\partial}{\partial x} f(x,y), \frac{\partial}{\partial y} f(x,y)\right).$$

In this case, the vector field F is perpendicular to the level sets of f.

For $f(x,y)$, choose a point (x_0, y_0) in the domain of f. Define the function $y(x)$ to be a level curve for f. Then

$$\begin{aligned} f(x,y(x)) &= c \\ \frac{d}{dx} f(x,y(x)) &= 0 \\ \frac{\partial}{\partial x} f(x,y(x)) + \frac{\partial}{\partial y} f(x,y(x)) \frac{dy}{dx} &= 0, \quad \text{by the chain rule} \end{aligned} \tag{2}$$

One identity of conservative field is that its curl is 0. To check this, note that

$$\begin{aligned} \nabla \times F(x,y) &= \frac{\partial}{\partial y} M(x,y) - \frac{\partial}{\partial x} N(x,y) \\ &= \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y) - \frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y) = 0 \end{aligned}$$

if are able to reverse the order of differentiation. Thus, if (1) is a conservative field,

$$\frac{\partial}{\partial y} M(x,y) = \frac{\partial}{\partial x} N(x,y). \tag{3}$$

To reverse the order of thinking, the differential equation,

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{4}$$

is called **exact** if the relation (3) holds. In this case, the vector field (1) is conservative and, thus, has a potential f. If we define $F(x,y) = \nabla f(x,y)$, then by (2), we have a solution to (4).

Exercise 1. *Show that*

$$a_1'(x)y + a_1(x)\frac{dy}{dx} = 0 \tag{5}$$

is exact.

Recall that we started with a linear differential equation,

$$\frac{dy}{dx} + P(x)y = Q(x)$$

and found an integrating factor $\mu(x)$ so that the equation had the form in (5).

2 Strategies for Solving Exact Differential Equations

For an exact first order differential equation, we have that

$$M(x,y) = \frac{\partial}{\partial x} f(x,y).$$

Thus, we integrate M with respect to x, leaving y variable,

$$f(x,y) = \int M(x,y)dx + g(y).$$

Here, the constant of integration is a function of y. Next note that

$$N(x,y) = \frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\int M(x,y)dx + g'(y).$$

We complete by identifying $g'(y)$ and integrating. Alternatively,

$$N(x,y) = \frac{\partial}{\partial y} f(x,y)$$

and we integrate N with respect to y, leaving x variable,

$$f(x,y) = \int N(x,y)dy + h(x).$$

with the constant of integration now a function of x. Also

$$M(x,y) = \frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\int M(x,y)dx + h'(x).$$

Finally, we compete the solution by identifying $h'(x)$ and integrating.

3 Examples

Example 2. *First we check that the differential equation*

$$6xy + 3(x^2 - y^2)\frac{dy}{dx} = 0, quady(0) = 1 \tag{6}$$

is exact.

$$M(x, y) = 6xy \quad \text{and} \quad N(x, y) = 3(x^2 - y^2)$$

The partial derivatives

$$\frac{\partial}{\partial y}M(x, y) = 6x \quad \text{and} \quad \frac{\partial}{\partial x}N(x, y) = 6x$$

show that the differential equation is exact. To find an implicit solution, we integrate M with respect to x

$$f(x, y) = \int 6xy\ dx = 3x^2y + g(y).$$

Then we differentiate to find N

$$N(x, y) = \frac{\partial}{\partial y}f(x, y) = 3x^2 + g'(y).$$

Next, the integratipn

$$g'(y) = -3y^2 \quad \text{and} \quad g(y) = -y^3$$

completes the implicit solution.

$$f(x, y) = 3x^2y - y^3 = c.$$

To determine c, use the initial condition

$$c = f(0, 1) = 3 \cdot 0^2 \cdot 1 - 1^3 = -1.$$

Exercise 3. *Find the solution to (6) by first integrating N.*

Example 4. *For*

$$(2xy - 9x^2) + (2y + x^2 + 1)\frac{dy}{dx} = 0, \quad y(0) = -3,$$

we have

$$M(x, y) = 2xy - 9x^2 \quad \text{and} \quad N(x, y) = 2y + x^2 + 1$$

The partial derivatives

$$\frac{\partial}{\partial y}M(x, y) = 2x \quad \text{and} \quad \frac{\partial}{\partial x}N(x, y) = 2x.$$

So the equation is exact. Next we integrate,

$$f(x, y) = \int (2xy - 9x^2)dx + g(y) = x^2y - 3x^2 + g(y).$$

We now find N in terms of this expression for f, i.e.,

$$N(x, y) = \frac{\partial}{\partial y}f(x, y) = x^2 + g'(y)$$

So,

$$g'(y) = 2y + 1 \quad and \quad g(y) = y^2 + y.$$

Together, we have

$$f(x, y) = \int (2xy - 9x^2)dx + g(y) = x^2y - 3x^2 + y^2 + y. \tag{7}$$

The general implicit solution is

$$x^2y - 3x^2 + y^2 + y. = c.$$

With the initial condition,

$$c = 0^2 \cdot (-3) + (-3) \cdot 0^2 - (-3)^2 + (-3) = 6.$$

We always have the alternative course.

$$f(x, y) = \int (2y + x^2 + 1)dy + g(y) = y^2 + x^2y + y + h(x)$$

$$M(x, y) = \frac{\partial}{\partial x} f(x, y) = 2xy + h'(x).$$

Returning to the expression for M, we find

$$h'(x) = -9x^2 \quad and \quad h(x) = -3x^2$$

yielding the same expression (7).

Exercise 5. *Use the quadratic formula, to find an explicit solution for the differential equation above. Use the initial condition the choose which of the two solutions from the quadratic formula solves the differental equation.*

Exercise 6. *Determine whether the equation is exact. If it is, then solve it.*

- $\left(3y - \frac{x}{y^2}\right) \frac{dy}{dx} + \frac{1}{y} = 0.$
- $2xy\frac{dy}{dx} + \cos(x) + y^2 = 0, y(1) = \pi.$

4 Special Integrating Factors

For linear equations, we were able to find an integrating factor μ by solving an auxillary differential equation. Because this equation could be solved by separation of variables, we could generally find μ. Sometimes a transformation of the variables results in a linear equation.

Other times, a similar approach can be considered for nonlinear equations that transform them into exact eqations. In this case the differential equation that defines the integrating factor may be harder to solve than the original differential equation. However, some special forms of this equation can be solved and these from the basis for the integrating factor.