

Moments and Generating Functions

September 24 and 29, 2009

Some choices of g yield a specific name for the value of $Eg(X)$.

1 Moments, Factorial Moments, and Central Moments

- For $g(x) = x$, we call EX the **mean** of X and often write μ_X or simply μ if only the random variable X is under consideration.

- S , the number of successes in n Bernoulli trials with success parameter p , has mean np .
- The mean of a geometric random variable with parameter p is $1/p$.
- The mean of an exponential random variable with parameter β is β .
- A standard normal random variable has mean 0.

- For $g(x) = x^m$, EX^m is called the **m -th moment** of X .

- If X is a Bernoulli random variable, then $X = X^m$. Thus, $EX^m = EX = p$.
- For a uniform random variable on $[0, 1]$, the m -th moment is $\int_0^1 x^m dx = 1/(m+1)$.
- The third moment for Z , a standard normal random, is 0. The fourth moment,

$$\begin{aligned} EZ^4 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^4 \exp\left(-\frac{z^2}{2}\right) dz = -\frac{1}{\sqrt{2\pi}} \left(z^3 \exp\left(-\frac{z^2}{2}\right) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} 3z^2 \exp\left(-\frac{z^2}{2}\right) dz \right) \\ &= 3EZ^2 = 3 \end{aligned}$$

$$\begin{aligned} u(z) &= z^3 & v(t) &= -\exp\left(-\frac{z^2}{2}\right) \\ u'(z) &= 3z^2 & v'(t) &= z \exp\left(-\frac{z^2}{2}\right) \end{aligned}$$

- For T , an exponential random variable, we integrate by parts

$$\begin{aligned} ET^m &= \int_0^{\infty} t^m \frac{1}{\beta} \exp-(t/\beta) dt = t^m \exp-(t/\beta) \Big|_0^{\infty} + \int_0^{\infty} mt^{m-1} \exp-(t/\beta) dt \\ &= \beta m \int_0^{\infty} t^{m-1} \frac{1}{\beta} \exp-(t/\beta) dt = m\beta ET^{m-1} \end{aligned}$$

$$\begin{aligned} u(t) &= t^m & v(t) &= \exp-(t/\beta) \\ u'(t) &= mt^{m-1} & v'(t) &= \frac{1}{\beta} \exp-(t/\beta) \end{aligned}$$

Thus, by induction, we have that

$$ET^m = \beta^m m!.$$

- If $g(x) = (x)_k$, where $(x)_k = x(x-1)\cdots(x-k+1)$, then $E(X)_k$ is called the **k -th factorial moment**.

– If X is uniformly distributed on $\{1, 2, \dots, n\}$, then

$$E(X)_k = \sum_{x=1}^n (x)_k \frac{1}{n} = \frac{1}{n} \cdot \frac{1}{k+1} \sum_{x=1}^n \Delta_+(x)_{k+1} = \frac{1}{n} \cdot \frac{(n+1)_{k+1}}{k+1}$$

- If $g(x) = (x - \mu)^k$, then $\mu_k = E(X - \mu)^k$ is called the **k -th central moment**.
- The second central moment $\sigma_X^2 = E(X - \mu)^2$ is called the **variance**. Note that

$$\text{Var}(X) = E(X - \mu)^2 = EX^2 - 2\mu EX + \mu^2 = EX^2 - 2\mu^2 + \mu^2 = EX^2 - \mu^2.$$

σ_X , the square root of the variance, is called the **standard deviation**.

The **standardized version** of X is

$$Z = \frac{X - \mu_X}{\sigma_X}.$$

It has mean 0 and variance 1. It is also called **z -value**, **z -score**, and **normal scores**. Note that

$$\text{Var}(aX + b) = E[(aX + b) - (a\mu_X + b)]^2 = E[(aX - a\mu_X)]^2 = a^2 E[(X - \mu_X)]^2 = a^2 \text{Var}(X)$$

and the standard deviation

$$\sigma_{aX+b} = |a|\sigma_X.$$

– If X is a Bernoulli random variable, the variance

$$\text{Var}(X) = EX^2 - \mu_X^2 = p - p^2 = p(1 - p).$$

– If U is uniform random variable on $[0, 1]$, then

$$\text{Var}(U) = EU^2 - \mu_U^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$$

– If X is uniformly distributed on $\{1, 2, \dots, n\}$, then

$$\begin{aligned} \text{Var}(X) &= EX^2 - \mu_X^2 = EX(X-1) + \mu_X - \mu_X^2 \\ &= \frac{1}{n} \cdot \frac{(n+1)n(n-1)}{3} + \frac{n+1}{2} - \frac{(n+1)^2}{4} \\ &= (n+1) \left(\frac{(n-1)}{3} + \frac{1}{2} - \frac{n+1}{4} \right) \\ &= \frac{n+1}{12} (4(n-1) + 6 - 3(n+1)) = \frac{n+1}{12} (n-1) = \frac{1}{12} (n^2 - 1). \end{aligned}$$

– If Z is a standard normal random, then $EZ = 0$, thus $\text{Var}(Z) = EZ^2 = 1$.

– For T , an exponential random variable,

$$\text{Var}(T) = ET^2 - (ET)^2 = 2\beta^2 - \beta^2 = \beta^2.$$

- The **skewness** of a random variable X

$$\alpha_3 = \frac{\mu_3}{\sigma^3},$$

is the third moment of the standardized version of X .

- The **kurtosis** of a random variable X compares the fourth moment of the standardized version of X to that of a standard normal random variable.

$$\alpha_4 = \frac{\mu_4}{\sigma^4} - 3.$$

2 Generating Functions

For generating functions, it is useful to recall that if h has a converging infinite Taylor series in a interval about the point $x = a$, then

$$h(x) = \sum_{n=0}^{\infty} \frac{h^{(n)}(a)}{n!} (x - a)^n$$

Where $h^{(n)}(a)$ is the n -th derivative of h evaluated at $x = a$.

- If $g(x) = \exp(i\theta x)$, then

$$\phi_X(\theta) = E \exp(i\theta X)$$

is called the **Fourier transform** or the **characteristic function**. Because $|g(x)| = |\exp(i\theta x)| = 1$, the expectation exists for any random variable.

- Similarly, $g(x) = \exp(tx)$, then

$$M_X(t) = E \exp(tX)$$

is called the **Laplace transform** or the **moment generating function**. To see the basis for this name, note that if we can reverse the order of differentiation and integration, then

$$\begin{aligned} \frac{d}{dt} M_X(t) &= EX \exp(tX) & \frac{d}{dt} M(0) &= EX \\ \frac{d^2}{dt^2} M_X(t) &= EX^2 \exp(tX) & \frac{d^2}{dt^2} M(0) &= EX^2 \\ &\vdots & &\vdots \\ \frac{d^k}{dt^k} M_X(t) &= EX^k \exp(tX) & \frac{d^k}{dt^k} M(0) &= EX^k \end{aligned}$$

- For a standard normal random variable Z ,

$$\begin{aligned} M_Z(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \exp\left(-\frac{z^2}{2}\right) dz = e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2 - 2tz + t^2}{2}\right) dz \\ &= e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(z-t)^2}{2}\right) dz = e^{t^2/2}. \end{aligned}$$

Thus,

$$M_Z(t) = \sum_{k=0}^{\infty} \frac{d^k}{dt^k} M(0) \frac{t^k}{k!} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!2^n} = e^{t^2/2}.$$

Thus, the odd moments of the standard normal is zero.

$$EZ^{2n} = \frac{(2n)!}{n!2^n}.$$

– For an exponential random variable T and for $t < 1/\beta$,

$$\begin{aligned} M_T(t) &= \int_0^\infty e^{tu} \frac{1}{\beta} \exp(-u/\beta) du = \int_0^\infty \frac{1}{\beta} \exp u(t - 1/\beta) du \\ &= \frac{1}{1 - \beta t} = \sum_{k=0}^\infty (\beta t)^k \end{aligned}$$

and the moments of T can be determined by examining the power series.

- The **cumulant generating function** is defined to be

$$K_X(t) = \log M_X(t).$$

The k -th terms in the Taylor series expansion at 0,

$$k_n(X) = \frac{1}{k!} \frac{d^k}{dt^k} K_X(0)$$

is called the **k -th cumulant**. For the first and second cumulant,

$$\begin{aligned} K'_X(t) &= \frac{M'_X(t)}{M_X(t)} & K'_X(0) &= M'_X(0) = EX \\ K''_X(t) &= \frac{M_X(t)M''_X(t) - M'_X(t)^2}{M_X(t)^2} & K''_X(0) &= M''_X(0) - M'_X(0)^2 = EX^2 - (EX)^2 = \text{Var}(X) \end{aligned}$$

- If X is \mathbb{Z}^+ -valued and $g(x) = z^x$, then

$$\rho_X(z) = Ez^X = \sum_{x=0}^\infty P\{X = x\}z^x = \sum_{x=0}^\infty f_X(x)z^x$$

is called the **(probability) generating function**. In the complex variable z , ρ is an analytic function. Its radius of convergence is at least 1 and thus we can obtain the derivatives of ρ by differentiating term by term. At $z = 0$ we obtain,

$$f_X(x) = \frac{1}{x!} \frac{d^x}{dz^x} \rho_X(0).$$

If we can take the derivative at $z = 1$, we have

$$\begin{aligned} \frac{d}{dz} \rho_X(z) &= EXz^{X-1} & \rho(1) &= EX \\ \frac{d^2}{dz^2} \rho_X(z) &= EX(X-1)z^{X-2} & \rho(1) &= E(X)_2 \\ & \vdots & & \vdots \\ \frac{d^k}{dz^k} \rho_X(z) &= E(X)_k z^{X-k} & \rho(1) &= E(X)_k \end{aligned}$$

– For X , a geometric random variable,

$$\rho_X(z) = Ez^X = \sum_{x=0}^\infty f_X(x)z^x = \sum_{x=1}^\infty p(1-p)^{x-1}z^x = \frac{pz}{1 - (1-p)z}.$$

$$\begin{aligned}\rho'_X(z) &= \frac{p}{(1-(1-p)z)^2} & \rho_X(1) &= \frac{1}{p} \\ \rho''_X(z) &= \frac{2p(1-p)}{(1-(1-p)z)^3} & \rho_X(2) &= \frac{2(1-p)}{p^2}\end{aligned}$$

and

$$\begin{aligned}\text{Var}(X) &= EX^2 - (EX)^2 = EX(X-1) + EX - (EX)^2 \\ &= \frac{2(1-p)}{p^2} + \frac{1}{p} - \left(\frac{1}{p}\right)^2 = \frac{2(1-p) + p - 1}{p^2} = \frac{1-p}{p^2}\end{aligned}$$

– For X , a binomial random variable based on n Bernoulli trials,

$$\rho_X(z) = Ez^X = \sum_{x=0}^n f_X(x)z^x = \sum_{x=0}^n \binom{n}{x} p^x (1-p)^{n-x} z^x = ((1-p) + zp)^n.$$

by the binomial theorem.

$$\rho''_X(z) = n(n-1)p^2((1-p) + zp)^{n-2} \quad EX(X-1) = \rho''_X(1) = n(n-1)p$$

and

$$\begin{aligned}\text{Var}(X) &= EX^2 - (EX)^2 = EX(X-1) + EX - (EX)^2 \\ &= n(n-1)p^2 + np - (np)^2 = np - np^2 = np(1-p).\end{aligned}$$

The moment generating function $M_X(t) = \rho_X(e^t)$.