The Mass-Spring Oscillator

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1 Deriving the Governing Equation

We being with **Newton's second law**

$$F = ma = m\frac{d^2y}{dt^2} = my''.$$

Hooke's law is a principle of physics that states that the force F needed to extend or compress a spring by some distance y is proportional to that distance and opposes the direction of the force.

$$F_{\text{spring}} = -ky.$$

The constant of proportionality k is called the **spring constant**.

Practically all mechanical systems also experience friction. Here, the force is typically modeled by a term proportional to velocity and again and opposes the direction of the force.

$$F_{\text{friction}} = -b\frac{dy}{dt} = -by'$$

The constant of proportionality b is called the **damping constant**.

Finally the spring may be subject to external forces like gravity or direct forcing. We will indicate this by F_{external}. Taken together, we have a second order linear ordinary differential equation

$$my'' + by' + ky = F_{\text{ext}}.$$

This is the differential equation that governs the motion of a mass-spring oscillator.

2 Behavior without Friction

To start, we consider on external force and no friction,

$$my'' + ky = 0.$$

Because this is meant to model the action of a spring, so we look for a solution of the form $y(t) = A\cos\omega t$ and look to determine the angular frequency, ω. Then,

$$mA\omega^2\cos\omega t - kA\cos\omega t = (m\omega^2 - k)cos\omega t.$$

Thus,

$$\omega^2 = k/m, \quad \omega = \sqrt{k/m}.$$

So the angular frequency,

- increases with k. Stiffer spring oscillate faster.
- decreases with m. More massive springs oscillate slower.

3 Behavior with Friction

3.1 Damped Oscillatory Behavior

If we add damping, then we have the differential equation.

$$my'' + by' + ky = 0. \tag{1}$$

If we assume that damping results in an exponential damping to the oscillator.

$$\begin{aligned} y(t) &= Ae^{-t}\cos t. \qquad (2)\\ y'(t) &= A(e^{-t}(-\sin t) + (-e^{-t}\cos t) = Ae^{-t}(-\sin t - \cos t)\\ y''(t) &= Ae^{-t}((-\cos t + \sin t) - (-\sin t - \cos t)) = Ae^{-t}(2\sin t) \end{aligned}$$

Take $m = 1$, $b = 2$, and $k = 2$.

$$\begin{array}{rlll} y'' & = Ae^{-t} & & (2\sin t)) \\ 2y' & = Ae^{-t} & ((-2\cos t)+ & (-2\sin t)) \\ 2y & = Ae^{-t} & (-2\cos t) & \end{array}$$

and, consequently,

$$y'' + 2y' + 2y = 0. \tag{3}$$

Thus, (2) is a solution to (3).

Notice that for the case of no damping ($b = 0$), a solution has frequency $\omega_0 = \sqrt{k/m} = \sqrt{2}$. This is greater that the frequency of the damped oscillator $\omega = 1$.

3.2 Overdamped Behavior

If we take $y(t) = Ae^{-t}$, then

$$\begin{array}{rl} y'' & = Ae^{-t} \\ 2y' & = -2Ae^{-t} \\ y & = Ae^{-t} \end{array}$$

and, we have a solution to (1) with $m = 1$, $b = 2$, and $k = 1$ that does not have any oscillatory behavior. Thus, in reducing the stiffness of the spring from $k = 2$ to $k = 1$, the friction force in more dominating and the spring no longer oscillates.

4 External Force

Let's return to the equation that models damped oscillatory behavior subject to a sinusoidal forcing, frequency γ

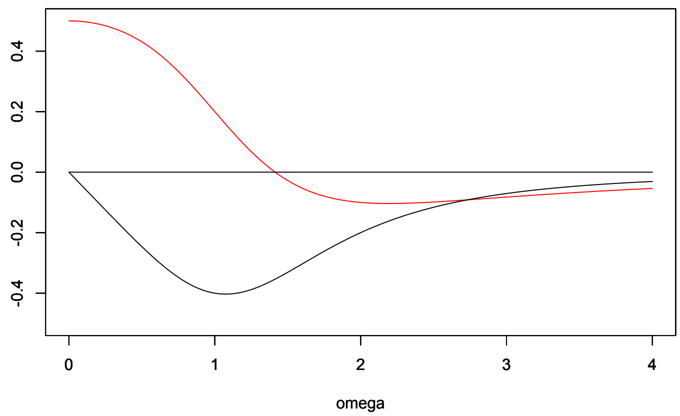

Figure 1: A (in black) and B (in red) as a function of γ.

$$y'' + 2y' + 2y = \sin \gamma t$$

and look for a solution

$$y = A \sin \gamma t + B \cos \gamma t$$

Then,

$$\begin{aligned} 2y(t) &= \quad 2A \sin \gamma t + 2B \cos \gamma t \\ 2y'(t) &= \quad 2A\gamma \cos \gamma t - 2B\gamma \sin \gamma t \\ y''(t) &= \quad -A\gamma^2 \sin \gamma t - B\gamma^2 \cos \gamma t \end{aligned}$$

Adding, we obtain

$$(2A - 2B\gamma - A\gamma^2) \sin \gamma t + (2B + 2A\gamma - B\gamma^2) \cos \gamma t = \sin \gamma t$$

Thus,

$$A(2 - \gamma^2) - 2B\gamma = 1 \quad \text{and} \quad B(2 - \gamma^2) + 2A\gamma = 0$$

Exercise 1. *Show that*

$$A = \frac{2 - \gamma^2}{(2 - \gamma^2)^2 + 4\gamma^2} \quad \textit{and} \quad B = \frac{-2\gamma}{(2 - \gamma^2)^2 + 4\gamma^2}$$

We will spend some time look at the behavior of second order linear ordinary differential equation with constant coefficients and then return to a more detailed analysis of the mass-spring oscillator.