Homogeneous Linear Equations

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1 Introduction

We now begin a careful examination of **homogeneous** second-order constant-coefficient differential equations

$$ay'' + by' + cy = 0 \tag{1}$$

The phrasing homogeneous refer to the fact that the right side of the equation is 0. In term of the mass-spring oscillator, we are examining the case of *no* forcing. In reference to this example, we will denote the independent variable by t.

The next exercise consists two quick observations.

Exercise 1.

- *The function $y(t) = 0$ is a solution to (1).*
- *If y_1 and y_2 are solutions to (1) and c_1 and c_2 are real numbers, then*

$$c_1 y_1(t) + c_2 y_2(t) \tag{2}$$

is a solution to (1).

The second item makes the important statement that a *linear combination* of solutions to a *linear differential equation* is also a solution.

We will begin by looking at solutions of the form $y(t) = e^{rt}$ and discover which values of r result in y being a solution to (1).

$$\begin{aligned} ay'' + by' + cy &= 0 \\ ar^2e^{rt} + bre^{rt} + ce^{rt} &= 0 \\ e^{rt}(ar^2 + br + cr) &= 0 \end{aligned}$$

Because e^{rt} is never 0, we find that the value r must be a solution to the quadratic equation, know here as the **auxiliary equation** or **characteristic equation**.

$$ar^2 + br + cr = 0 \tag{3}$$

From the quadratic formula, we have to solutions (or roots).

$$r_+ = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_- = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We have three classes of solutions depending on the discriinant

$$d = b^2 - 4ac.$$

1. The roots are real and distinct ($d > 0$).
2. The roots are repeated ($d = 0$).
3. The roots are complex and distinct ($d < 0$).

2 Real Distinct Roots to the Auxilliary Equation

Example 2. *For the differential equation*

$$2y'' + 5y' + 3y = 0, \tag{4}$$

we have the auxiliary equation

$$\begin{aligned} 2r^2 + 5r + 3 &= 0 \\ (2r - 1)(r + 3) &= 0 \end{aligned}$$

Thus we have the two roots

$$r_+ = \frac{1}{2} \quad \text{and} \quad r_- = -3.$$

Consequently, both

$$y_1(t) = e^{t/2} \quad \text{and} \quad y_2(t) = e^{-3t}$$

are solutions to (4). Because the differential equation is linear, we have for any real numbers c_1 and c_2,

$$c_1 e^{t/2} + c_2 e^{-3t}$$

is a solution to (4).

3 Properties of the Solutions

Based on this example, let's gather some of the properties of homogeneous second-order constant-coefficient differential equations.

- Given an initial position $y(t_0) = y_0$ and an initial slope $y'(t_0) = y_1$, these differential equations have a unique solution for any real value t. This is the standard formulation for an **initial value problem**.
- We call the functions $y_1, y_2, \ldots, y_k$ **linearly independent** if no linear combination of these equations is equal to the zero equation.
- If y_1 and y_2 are linearly independent solutions on $\mathbb{R}$, then we can find values c_1 and c_2 so that $c_1 y_1(t) + c_2 y_2(t)$ solves the initial value problem. This amounts to solving two equations (one for y_0, one for y_1) with two unknowns (c_1 and c_2).

Exercise 3. *Show that an equivalent definition of linear independence is that no one of functions can be written as a linear combination of the others. For the case $k = 2$ this amounts to saying that the two functions y_1 and y_2 are not constant multiples of each other.*

4 Real Repeated Roots

If the auxillary equation has the discriminant $d = 0$, then $r = r_- = r_+$. The existence/uniqueness properties for the solution that we can find a second solution that is linearly independent of $y_1(t) = e^{rt}$.

Example 4. *For the differential equation*

$$y'' - 6y' + 9y = 0 \tag{5}$$

we have the auxiliary equation

$$r^2 - 6r + 9 = (r-3)^2$$

we can easily verify that $y_1(t) = e^{3t}$ is a solution. We will now check that $y_2(t) = te^{3t}$ is also a solution. Notice that y_1 and y_2 are linearly independent.

$$\begin{array}{rlcll} y_2(t) = & te^{3t} & = & & te^{3t} \\ y_2'(t) = & -3te^{3t} + e^{3t} & = & e^{3t} & -3te^{3t} \\ y_2''(t) = & 9te^{3t} + 3e^{3t} + 3e^{3t} & = & 6e^{3t} & +9te^{3t} \end{array}$$

Thus,

$$\begin{array}{rll} 9y_2(t) = & & +9te^{3t} \\ -6y_2'(t) = & -6e^{3t} & -18te^{3t} \\ y_2''(t) = & +6e^{3t} & +9te^{3t} \end{array}$$

If we add the two sides of this equation, we find a solution to (5),

Thus, we have the general solution

$$y(t) = c_1e^{3t} + c_2te^{3t}$$

to (5).

For the initial values $y(0) = 1, y'(0) = -2$, we have

$$\begin{array}{rcl} y(0) & = & c_1 \\ y'(t) & = & 3c_1e^{3t} + c_2(e^{3t} + 3te^{3t}) \\ y'(0) & = & 3c_1 + c_2 \end{array}$$

Thus, $c_1 = 1$ and

$$-2 = 3c_1 + c_2, \quad -2 = 3 + c_2, \quad c_2 = -5.$$