Families of Continuous Distributions

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We shall, in general, denote the density of a parametric family of discrete distributions by $f_X(x|\theta)$ for the distribution depending on the parameter θ. Some of the mystery surrounding these densities will be solved when we begin to look at multiple random variables.

1 Uniform Distributions

A **continuous uniform distribution** on $[a, b]$ has density

$$f_X(x|a,b) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

$$EX = \frac{a+b}{2}, \quad \text{Var}(X) = \frac{(b-a)^2}{12}, \quad M_X(t) = \frac{1}{(b-a)t}(e^{tb} - e^{ta}).$$

2 Exponential Distributions

An **exponential distribution** with parameter β has density

$$f_X(x|a,b) = \begin{cases} \frac{1}{\beta} e^{-t/\beta} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$EX = \beta, \quad \text{Var}(X) = \beta^2, \quad M_X(t) = \frac{1}{1-\beta t}.$$

The exponential distribution also has the **memoryless property**. In symbols,

$$P\{X > t + s | X > t\} = P\{X > s\}.$$

The probability that an individual survives s additional time units does not depend on the present age of the individual.

Exercise 1. *Prove the memoryless property of the exponential random variable.*

The **double exponential distribution** can be obtained by shifting and symmetrizing the exponential distribution. The density function is

$$f_X(x|\mu,\beta) = \frac{1}{2\beta} \exp -\frac{|x-\mu|}{\beta}, \quad x \in \mathbb{R}.$$

By symmetry $EX = \mu$. Check that $\text{Var}(X) = 2\beta^2$.

3 Gamma Distributions

The **Gamma function** is defined for $\alpha > 0$ by

$$\Gamma(\alpha) = \int_0^\infty t^\alpha e^{-t} \, \frac{dt}{t}.$$

Using integration by parts, one can show that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Note that $\Gamma(1) = 1$. For the case that α is a non-negative integer, we can use induction to see that $\Gamma(\alpha+1) = \alpha!$

Because the integrand in the gamma function is non-negative, we have a density function for a continuous random variable,

$$f_T(t|\alpha) = \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-t} & \text{for } 0 \le t, \\ 0 & \text{otherwise} \end{cases}$$

Let $X = \beta T$ to obtain the two parameter family

$$f_X(x|\alpha,\beta) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} & \text{for } 0 \le x, \\ 0 & \text{otherwise.} \end{cases}$$

The gamma densities with $\alpha = 1, 2, \cdots, 6$, and $\beta = 1$ are graphed using R in Figure 1 using

```
> curve(dgamma(x,1,1),0,10)
> for (i in 2:6){curve(dgamma(x,i,1),0,10,add=TRUE)}
```

The α parameter determine the sharpness of the peak and is called the **shapeness parameter**. The β parameter determine the spread of the distribution and is called the **scale parameter**.

$$ET = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha e^{-t} \, dt = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha.$$

Thus,

$$EX = E[\beta T] = \alpha\beta.$$

In addition,

$$ET^2 = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha+1} e^{-t} \, dt = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)} = (\alpha+1)\alpha.$$

Thus,

$$\text{Var(T)} = ET^2 - (ET)^2 = (\alpha+1)\alpha - \alpha^2 = \alpha$$

and

$$\text{Var}(X) = \text{Var}(\beta T) = \beta^2 \text{Var}(T) = \alpha\beta^2.$$

The moment generating function for T, change variables, setting $u = t(1-s)$

$$M_T(s) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha e^{-t} e^{st} \, \frac{dt}{t} = \frac{1}{\Gamma(\alpha)} (1-s)^{-\alpha} \int_0^\infty u^\alpha e^{-u} \, \frac{du}{u} = (1-s)^{-\alpha}.$$

Then,

$$M_X(r) = E[\exp rX] = E[\exp r\beta T] = M_T(\beta r) = (1-\beta r)^{-\alpha}.$$

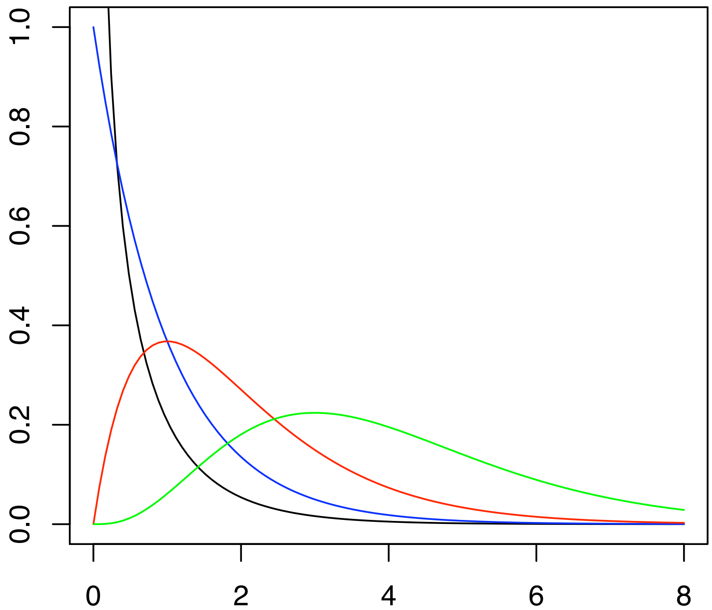

Figure 1: Gamma density with $\alpha = 1, 2, 3, 4, 5, 6$ and $\beta = 1$.

4 Chi-square Distributions

The **chi-square** densities result from taking $\alpha = p/2$ for some non-negative integer p (called the **degrees of freedom**) and $\beta = 2$. Then, the density becomes

$$f_X(x|p) = \begin{cases} \frac{1}{\Gamma(p/2)2^{p/2}} x^{(p/2)-1} e^{-x/2} & \text{for } 0 \le x, \\ 0 & \text{otherwise.} \end{cases}$$

$EX = p/2$, $\mathrm{Var}(X) = 2p$, and $M_X(t) = (1-2t)^{-p/2}$. We shall later learn the relationship between the chi-square distribution and the normal distribution.

5 Beta Distributions

The **beta distribution** has two parameters α and β. It is based on the **Beta function**

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt$$

. The density is concentrated on $[0, 1]$.

$$f(x|\alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}.$$

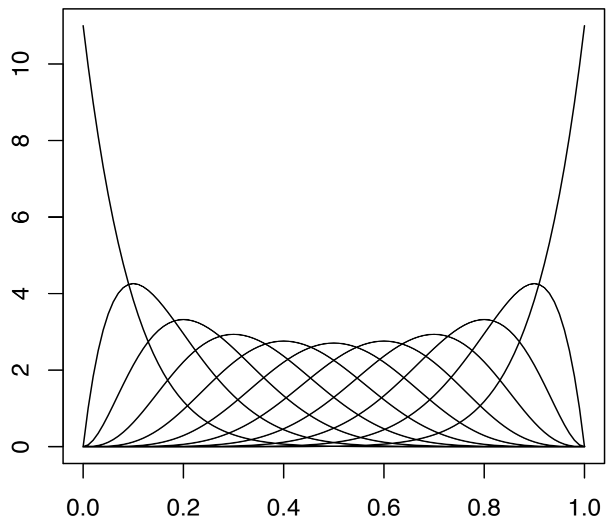

Figure 2: Beta density with $\alpha = 0, 1, \cdots, 10$ and $\beta = 12 - \alpha$.

$$EX = \frac{\alpha}{\alpha+\beta}, \quad \mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

The beta densities with $\alpha + \beta = 12$ are graphed using R in Figure 2 using

```
> curve(dbeta(x,1,11),0,1)
> for (i in 2:11){curve(dbeta(x,i,12-i),0,1,add=TRUE)}
```

6 Normal Distributions

X is called a **normal random variable** if it is a linear function of Z a standard normal random variable.

$$X = \sigma Z + \mu, \qquad Z = \frac{X-\mu}{\sigma}$$

Its density

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp -\frac{(x-\mu)^2}{2\sigma^2}.$$

If Z has moment generating function M_Z then

$$M_X(t) = E[\exp tX] = E[\exp t(\sigma Z + \mu)] = e^{\mu t} M_Z(\sigma t).$$

Thus, we have that

$$EX = \mu, \quad \mathrm{Var}(X) = \sigma^2, \quad M_X(t) = \exp\left(\frac{\sigma^2}{2}t^2 + \mu t\right).$$

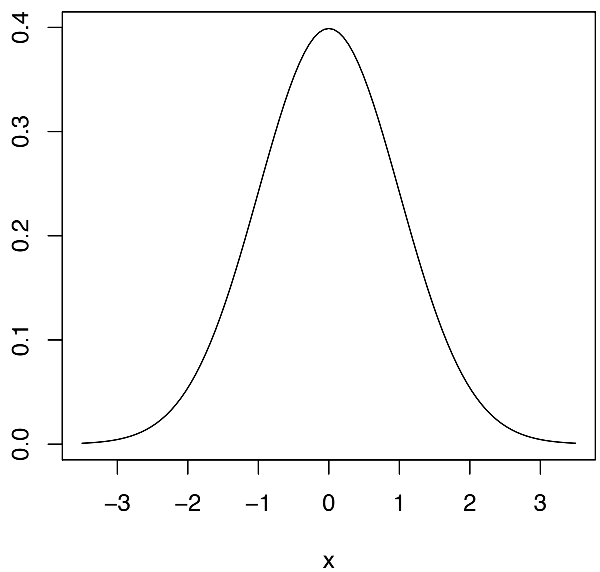

Figure 3: Standard normal density from R command `curve(dnorm(x,0,1),-3.5,3.5)`

7 Cauchy Distributions

Recall that

$$\frac{d}{dx}\tan^{-1}(x-\theta) = \frac{1}{1+(x-\theta)^2}.$$

Thus,

$$f_X(x|\theta) = \frac{1}{\pi}\frac{1}{1+(x-\theta)^2}, \quad x \in \mathbb{R}$$

is a valid probability density function.

The absolute mean $E|X|$ does not exist for the Cauchy distribution. To see this note that

$$\frac{1}{\pi}\int_0^b \frac{x}{1+(x-\theta)^2}\,dx = \frac{1}{2\pi}\log(1+(x-\theta)^2)\Big|_0^b = \frac{1}{2\pi}(\log(1+(b-\theta)^2) - \log(1+\theta^2)) \to \infty$$

as $b \to \infty$.