

Auxiliary Equations with Complex Roots

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For homogeneous second-order constant-coefficient differential equations,

$$ay'' + by' + cy = 0, \tag{1}$$

we focused our analysis on the auxiliary equation

$$ar^2 + br + cr = 0. \tag{2}$$

In particular, we considered the discriminant $d = b^2 - 4ac$.

In the previous section we considered the cases $d > 0$ and $d = 0$. Now, we turn to the one remaining case $d < 0$ where the roots of (2), r_- and r_+ are complex and distinct. In particular, the roots are

$$r_- = \alpha - i\beta \quad \text{and} \quad r_+ = \alpha + i\beta.$$

where

$$\alpha = -\frac{b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}.$$

If we follow the same procedure, then we would say that the solutions to (1) are

$$y_1(t) = e^{(\alpha+i\beta)t} \quad \text{and} \quad y_2(t) = e^{(\alpha-i\beta)t}$$

1 Introduction to Exponentials and Complex Numbers

To make sense of e to a complex power, we define

$$e^{\lambda+i\theta} = e^\lambda(\cos \theta + i \sin \theta) \tag{3}$$

To check that (3) has some of the expected properties, let

$$\zeta_1 = \lambda_1 + i\theta_1 \quad \text{and} \quad \zeta_2 = \lambda_2 + i\theta_2.$$

We would like to say that

$$e^{\zeta_1} e^{\zeta_2} = e^{\zeta_1 + \zeta_2}.$$

Let's check this

$$\begin{aligned}
e^{\zeta_1} e^{\zeta_2} = e^{\lambda_1 + i\theta_1} e^{\lambda_2 + i\theta_2} &= (e^{\lambda_1} (\cos \theta_1 + i \sin \theta_1))(e^{\lambda_2} (\cos \theta_2 + i \sin \theta_2)) \\
&= e^{\lambda_1} e^{\lambda_2} (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\
&\quad \text{(Multiplication is commutative.)} \\
&= e^{\lambda_1 + \lambda_2} (\cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i^2 \sin \theta_1 \sin \theta_2) \\
&\quad \text{(Properties of real exponents and complex multiplication.)} \\
&= e^{\lambda_1 + \lambda_2} ((\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)) \\
&\quad (i^2 = -1) \\
&= e^{\lambda_1 + \lambda_2} ((\cos(\theta_1 + \theta_2) + i(\sin(\theta_1 + \theta_2))) \\
&\quad \text{(Identities for the sine and cosine of a sum of angles.)} \\
&= e^{(\lambda_1 + i\theta_1) + (\lambda_2 + i\theta_2)} = e^{\zeta_1 + \zeta_2}. \\
&\quad \text{(The definition in (3).)}
\end{aligned}$$

Next, we take a derivative

$$\begin{aligned}
\frac{d}{dt} e^{i(\alpha + i\beta)t} &= \frac{d}{dt} (e^{\alpha t} (\cos \beta t + i \sin \beta t)) \\
&= \alpha e^{\alpha t} (\cos \beta t + i \sin \beta t) + \beta e^{\alpha t} (-\sin \beta t + i \cos \beta t) \\
&= e^{\alpha t} (\alpha (\cos \beta t + i \sin \beta t) + i\beta (\alpha (\cos \beta t + i \sin \beta t))) \\
&= (\alpha + i\beta) e^{\alpha t} (\cos \beta t + i \sin \beta t) = (\alpha + i\beta) e^{i(\alpha + i\beta)t}
\end{aligned}$$

as expected.

Because cosine is even and sine is odd,

$$\cos(-\theta) = \cos \theta \quad \text{and} \quad \sin(-\theta) = -\sin \theta.$$

Exercise 1. Show that

$$\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}). \quad \text{and} \quad \sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

So both $\cos \theta$ and $\sin \theta$ are linear combinations of $e^{i\theta}$ and $e^{-i\theta}$ and both $e^{i\theta}$ and $e^{-i\theta}$ are linear combinations of $\cos \theta$ and $\sin \theta$

2 Example

Example 2. For

$$y'' + y = 0$$

, the auxiliary equation

$$r^2 + 1 = 0$$

has roots

$$r_- = -i \quad \text{and} \quad r_+ = i$$

The general solution is

$$y(t) = c_1 \cos t + c_2 \sin t$$

If $y(0) = 3$ and $y'(0) = -4$, then

$$\begin{aligned} 3 = y(0) &= c_1 \\ y'(t) &= -c_1 \sin t + c_2 \cos t \\ -4 = y'(0) &= c_2 \end{aligned}$$

Thus,

$$y(t) = 3 \cos t - 4 \sin t$$

Exercise 3. For the general solution to a frictionless spring

$$my'' + ky = 0$$

What do the values c_1 and c_2 represent.

Example 4. Returning to the damped oscillator with $m = 1, b = 2$, and $k = 2$., we have the governing equation

$$y'' + 2y' + 2y = 0. \tag{4}$$

The auxiliary equation is

$$r^2 + 2r + 2 = 0$$

with roots

$$r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2} = \frac{-2 \pm i\sqrt{4}}{2} = -1 \pm i.$$

Thus, the two roots are

$$r_- = -1 - i \quad \text{and} \quad r_+ = -1 + i.$$

At this point, we have two choices. We can write the solutions as

$$y_1(t) = e^{(-1-i)t} = e^{-t}(\cos t - i \sin t) \quad \text{and} \quad y_2(t) = e^{(-1+i)t} = e^{-t}(\cos t + i \sin t) \tag{5}$$

with a general solution

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

or

$$\begin{aligned} \tilde{y}_1(t) &= e^{-t} \cos t \quad \text{and} \quad \tilde{y}_2(t) = e^{-t} \sin t \\ y(t) &= \tilde{c}_1 \tilde{y}_1(t) + \tilde{c}_2 \tilde{y}_2(t). \end{aligned} \tag{6}$$

We know from the exercise above that we can write the solutions in (5) as a linear combination of the solutions in (6) and vice versa.

For $y(0) = 0$ and $y'(0) = -3$, then

$$\begin{aligned} 0 = y(0) &= c_1 + c_2 \\ y'(t) &= c_1(-1-i)e^{(-1-i)t} + c_2(-1+i)e^{(-1+i)t} \\ -3 = y'(0) &= c_1(-1-i) + c_2(-1+i) \end{aligned}$$

So $c_2 = -c_1$,

$$-3 = c_1(-1 - i) - c_1(-1 + i) = -2ic_1, \quad \text{and} \quad c_1 = \frac{-3}{-2i} = -\frac{3i}{2}.$$

Thus,

$$\begin{aligned} y(t) &= -\frac{3i}{2}(e^{-t}(\cos t - i \sin t) - e^{-t}(\cos t + i \sin t)) \\ &= -\frac{3i}{2}(e^{-t}(-2i \sin t)) = -3e^{-t} \sin t. \end{aligned}$$

$$\begin{aligned} 0 = y(0) &= \tilde{c}_1 \\ y'(t) &= \tilde{c}_1 e^{-t}(-\sin t - \cos t) + \tilde{c}_2 e^{-t}(\cos t - \sin t) \\ -3 = y'(0) &= -\tilde{c}_1 + \tilde{c}_2 \end{aligned}$$

Thus, $\tilde{c}_1 = 0$, $\tilde{c}_2 = -3$, and

$$y(t) = -3e^{-t} \sin t.$$

Exercise 5. *Solve*

$$y'' - 6y' + 13y = 0$$

3 Returning to the Mass-Spring Oscillator

Returning to the mass-spring oscillator, we have the governing equation

$$my'' + by' + ky = 0$$

3.1 Frictionless Oscillator

For the case $b = 0$, we have the auxiliary $mr^2 + k = 0$ with roots $r_{\pm} = \pm\sqrt{k/m} = \omega_0$. The general solution is

$$y(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t. \tag{7}$$

To place this expression in a different form, recall the sum on sines formula

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2.$$

Now, set

- $c_1 = A \sin \phi$, $c_2 = A \cos \phi$
then
- $A = \sqrt{c_1^2 + c_2^2}$, $\phi = \tan(c_1/c_2)$.
With
- $\theta_1 = \phi$ and $\theta_2 = \omega_0 t$

we can rewrite (7)

$$y(t) = A \sin(\omega t + \phi)$$

(Tangent has period π and so the choice of ϕ depends on the signs of c_1 and c_2 .) In this expression A is called the **amplitude** of the oscillator and ϕ is called the phase.

3.2 Underdamped oscillatory motion

With $b \neq 0$ we have the roots of the auxiliary equation

$$r_- = \frac{-b - \sqrt{b^2 - 4mk}}{2m} \quad \text{and} \quad r_+ = \frac{-b + \sqrt{b^2 - 4mk}}{2m}.$$

If $4mk > b^2$, then the expression in the radical is negative. We write the roots

$$r_- = \frac{-b}{2m} - i\sqrt{\omega_0^2 - \frac{b^2}{4m^2}} \quad \text{and} \quad r_+ = \frac{-b}{2m} + i\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}. \quad (8)$$

In this case, the spring is oscillating, but damped. Thus, a spring is more likely to oscillate with a higher mass and a stiffer spring.

The damping is exponential with rate $-b/2m$. So the damping rate increases with higher friction or lower mass. The frequency of oscillator ω satisfies

$$\omega^2 = \omega_0^2 - \frac{b^2}{4m^2},$$

which is slower than the frictionless oscillator frequency, ω_0 . Now we can write the general solution as

$$y(t) = e^{-kt/m}(c_1 \cos \omega t + c_2 \sin \omega t)$$

As with the frictionless case, we can write

$$y(t) = Ae^{-kt/m} \sin(\omega t + \phi). \quad (9)$$

3.3 Damped motion

If $4mk > b^2$ then the roots (8) to the auxiliary equation are both real and negative.

Exercise 6. Check that both of the roots, r_- and r_+ , are negative.

Thus,

$$y(t) = c_1 e^{r_- t} + c_2 e^{r_+ t}$$

and the spring does not oscillate. It does not even move farther from equilibrium than its initial position. Write

$$y(t) = (c_1 e^{(r_- - r_+)t} + c_2) e^{r_+ t}$$

Because $r_- - r_+ < 0$, the long term behavior is determined by the smaller rate, r_+ .

3.4 Critically Damped

For the case in which $4mk = b^2$, then the auxiliary equation is a perfect square and, consequently, the roots $r_c = -2b/m$ are repeated. In this case the general solution

$$y(t) = e^{-kt/m}(c_1 + c_2 t)$$

and the spring can move initially away from the equilibrium. It still does not oscillate

Exercise 7. Find the maximum distance that the spring is from equilibrium. When does that take place?

3.5 Forced Oscillations

We will examine external forces to the mass-spring oscillator through a sinusoidal force. Thus, the governing equations

$$my'' + by' + ky = F_0 \cos \gamma t$$

. We will analyze the forcing using the solution (9) to the homogeneous equation. Using the method of undetermined coefficients,

$$y_p(t) = A_1 \cos \gamma t + A_2 \sin \gamma t.$$

So,

$$\begin{aligned} y_p(t) &= A_1 \cos \gamma t && + A_2 \sin \gamma t \\ y_p'(t) &= \gamma A_2 \cos \gamma t && - \gamma A_1 \sin \gamma t \\ y_p''(t) &= -\gamma^2 A_1 \cos \gamma t && + -\gamma^2 A_2 \sin \gamma t \\ ky_p(t) &= kA_1 \cos \gamma t && + kA_2 \sin \gamma t \\ by_p'(t) &= b\gamma A_2 \cos \gamma t && - b\gamma A_1 \sin \gamma t \\ my_p''(t) &= -m\gamma^2 A_1 \cos \gamma t && - m\gamma^2 A_2 \sin \gamma t \end{aligned}$$

Substituting into the governing equation, we find that

$$((k - m\gamma^2)A_1 + b\gamma A_2) \cos \gamma t + (-b\gamma A_1 + (k - m\gamma^2)A_2) \sin \gamma t = F_0 \cos \gamma t.$$

By equating the coefficients of $\cos \gamma t$ and $\sin \gamma t$, we have

$$(k - m\gamma^2)A_1 + b\gamma A_2 = F_0 \quad \text{and} \quad -b\gamma A_1 + (k - m\gamma^2)A_2 = 0. \quad (10)$$

Exercise 8. The solutions for A_1 and A_2 for (10) is

$$A_1 = \frac{F_0(k - m\gamma^2)}{(k - m\gamma^2)^2 + b^2\gamma^2} \quad \text{and} \quad A_2 = \frac{F_0 b\gamma}{(k - m\gamma^2)^2 + b^2\gamma^2}$$

This give us a particular solution

$$\begin{aligned} y_p(t) &= \frac{F_0}{(k - m\gamma^2)^2 + b^2\gamma^2} ((k - m\gamma^2) \cos \gamma t + b\gamma \sin \gamma t) \\ &= \frac{F_0}{(k - m\gamma^2)^2 + b^2\gamma^2} \left(\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2} \sin(\gamma t + \phi_f) \right) \\ &= \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}} \sin(\gamma t + \phi_f) \\ &= F_0 M(\gamma) \sin(\gamma t + \phi_f) \end{aligned}$$

where $\tan \phi_f = A_1/A_2 = (k - m\gamma^2)/(b\gamma)$ and $M(\gamma)$ is called the **frequency response curve** or **response curve**.

Exercise 9. In the underdamped case, M takes on its maximum at

$$\gamma_r = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$$

with value

$$M(\gamma_r) = \frac{1}{b\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}}.$$