# Exponential Families of Random Variables

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For this section, the parameter $\theta \in \mathbb{R}^k$ can be vector valued. A family of continuous (discrete) random variables is called an **exponential family** if the probability density functions (probability mass functions) can be expressed in the form

$$f_X(x|\theta) = h(x)c(\theta)\exp\left(\sum_{i=1}^{k}\theta_i t_i(x)\right).. \tag{1}$$

for $x$ in the common domain of the $f_X(x|\theta)$.

Obviously $h$ and $c$ are non-negative functions. The $t_i(x)$ are real-valued functions of the observations.

We may initially write the density functions using another parameterization $\eta$ and we use a mapping $\eta \mapsto \theta$ to put the density into the form seen in (1). In this form, $\theta$ is called the **natural parameter**.

**Example 1.** *Fix a value for $r$. For $X$ a negative binomial random variable,*

$$f_X(x|p) = \binom{x+r-1}{x} p^r(1-p)^x.$$

*Let $\theta = \log(1-p)$, and $t(x) = x$, then we can write this expression*

$$f_X(x|\theta) = \binom{x+r-1}{x}(1-e^\theta)^r \exp(\theta x).$$

**Example 2.** *For $X$ a gamma random variable,*

$$f_X(x|\alpha,\beta) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta} & \textit{for } 0 \le x, \\ 0 & \textit{otherwise.} \end{cases}$$

*Now let $\theta_1 = \alpha$ and $\theta_2 = -1/\beta$, $t_1(x) = \log x$ and $t_2(x) = x$, then for $x \ge 0$*

$$f_X(x|\theta) = x^{-1}\frac{(-\theta_2)^{\theta_1}}{\Gamma(\theta_1)}\exp\left(\theta_1 \log x + \theta_2 x\right).$$

**Example 3.** *For $X$ a normal random variable,*

$$f_X(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp -\frac{(x-\mu)^2}{2\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\mu^2/2\sigma^2}\exp\left(\frac{-x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2}\right)$$

*Here $\theta_1 = \mu/2\sigma^2$ and $\theta_2 = 1/\sigma^2$, $t_1(x) = x$ and $t_2(x) = x^2$ puts the normal distribution in the form that shows it to be an exponential family. Here*

$$c(\theta) = \frac{\theta_2^{1/2}}{\sqrt{2\pi}}\exp(-\theta_1).$$

Note that for continuous random variables

$$1 = c(\theta) \int h(x) \exp\left(\sum_{i=1}^{k} \theta_i t_i(x)\right) dx.$$

Thus,

$$0 = \frac{\partial c(\theta)}{\partial \theta_i} \int h(x) \exp\left(\sum_{i=1}^{k} \theta_i t_i(x)\right) dx + c(\theta) \int t_i(x) h(x) \exp\left(\sum_{i=1}^{k} \theta_i t_i(x)\right) dx,$$

$$0 = \frac{1}{c(\theta)} \frac{\partial c(\theta)}{\partial \theta_i} + E t_i(X),$$

and

$$-\frac{\partial}{\partial \theta_i} \log c(\theta) = E t_i(X).$$

**Example 4.** *For $X$ a negative binomial random variable, $\theta = \log(1-p)$*

$$c(\theta) = (1 - e^{\theta})^r, \quad t(x) = x.$$

$$\log c(\theta) = r \log(1 - e^{\theta})$$

$$EX = -\frac{\partial}{\partial \theta_i} \log c(\theta) = \frac{r e^{\theta}}{1 - e^{\theta}} = \frac{r(1-p)}{1-(1-p)} = r\left(\frac{1}{p} + 1\right)$$

**Example 5.** *For $X$ a gamma random variable,*

$$c(\theta) = \frac{(-\theta_2)^{\theta_1}}{\Gamma(\theta_1)}.$$

$$\log c(\theta) = \theta_1 \log(-\theta_2) + \log \Gamma(\theta_1).$$

*Then*

$$E[\log X] = -\frac{\partial}{\partial \theta_1} \left(\theta_1 \log(-\theta_2) + \log \Gamma(\theta_1)\right) = -\log(-\theta_2) - \frac{d}{d\theta_1} \log \Gamma(\theta_1) = \log(\beta) - \frac{d}{d\alpha} \log \Gamma(\alpha),$$

*and*

$$EX = -\frac{\partial}{\partial \theta_2} \left(\theta_1 \log(-\theta_2) + \log \Gamma(\theta_1)\right) = -\frac{\theta_1}{\theta_2} = \alpha\beta$$

**Exercise 6.** *Find $EX$ and $EX^2$ for the normal distribution using $c(\theta)$.*

**Exercise 7.** *For a one parameter exponential family,*

$$f_X(x|\theta) = h(x) c(\theta) \exp\left(\theta t(x)\right),$$

*use the ideas of the cumulant generating function to find* $\mathrm{Var}(t(X))$.