Nonhomogeneous Equations and Variation of Parameters

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1 Nonhomogeneous Equations

1.1 Review of First Order Equations

If we look at a first order *homogeneous* constant coefficient ordinary differential equation

$$by' + cy = 0.$$

then the corresponding auxiliary equation

$$ar + c = 0$$

has a root $r_1 = -c/a$ and we have a solution

$$y_h(t) = ce^{r_1 t} = c_1 e^{-ct/a}$$

If the equation is nonhomogeneous

$$by' + cy = f.$$

Then, we introduce the integrating factor $e^{ct/b}$

$$\begin{aligned} \frac{d}{dt}(e^{ct/b}y) &= e^{ct/b}f \\ e^{ct/a}y(t) &= c_1 + \int e^{ct/a}f(t)dt \\ y(t) &= c_1 e^{-ct/b} + e^{-ct/b}\int e^{ct/b}f(t)dt \\ y(t) &= y_h(t) + y_p(t) \end{aligned}$$

The solution is a sum of

- $y_h(t)$, the solution to the homogeneous equation. ($by_h' + cy_h = 0$). It has the constant that will be determined by the initial condition.
- $y_p(t)$, a solution that involves f.

Then,

$$b(y_h + y_p)' + c(y_h + y_p) = (by_h' + cy_h) + (by_p' + cy_p) = 0 + f = f.$$

We next take an similar, but less formal approach to second order equations, writing,

$$y = y_h + y_p$$

where y_h is a general solution to

$$ay_h'' + by_h' + cy_h = 0.$$

and y_p is a particular solution to

$$ay_p'' + by_p' + cy_p = f.$$

1.2 Examples

We gain intuition in the nature of particular solution through some illustrative exampels

Example 1. *For*

$$y'' + y' + 4y = 2t,$$

we try a particular solution $y_p(t) = At + B$*. Then*

$$\begin{aligned} y_p(t) &= \quad At \quad +B \\ y_p'(t) &= \quad A \\ y_p''(t) &= \quad 0 \end{aligned}$$

$$\begin{aligned} 4y_p(t) &= \quad 4At \quad +4B \\ y_p'(t) &= \quad A \\ y_p''(t) &= \quad 0 \end{aligned}$$

Thus,

$$4At + (A + 4B) = 2t$$

$$4A = 2, \quad A + 4B = 0$$

$$A = \frac{1}{2} \quad B = -\frac{1}{8}$$

and

$$y_p(t) = \frac{1}{3}t - \frac{1}{8}.$$

we can turn this suggestion into a strategy for the case that f is a polynomial. If the degree is m, then we will look for a particular solution

$$y_p(t) = A_m x^m + \cdots + A_1 x + A_0$$

that is also a polynomial of degree m. If this case the expression

$$ay_p'' + by_p' + cy$$

is also a polynomial of degree m. The **undetermined coefficients** $A_0, A_1, \ldots, A_m$ are selected so that the coefficients of $1, t, \ldots, t^m$ to match those of f. This gives us $m + 1$ equation in m unknowns. This technique is called the bf method of undetermined coefficients.

Example 2. *For the differential equation*

$$4y'' - 3y' - y = -2\cos t,$$

we note that the derivatives of sine will introduce the cosine. Thus, we search for a particular solution

$$y_p(t) = A\cos t + B\sin t$$

with undetermined coefficients A and B.

$$\begin{array}{rll} y_p(t) = & A\cos t & +B\sin t \\ y_p'(t) = & B\cos t & -A\sin t \\ y_p''(t) = & -A\cos t & -B\sin t \end{array}$$

$$\begin{array}{rll} -y_p(t) = & -A\cos t & -B\sin t \\ -3y_p'(t) = & -3B\cos t & +3A\sin t \\ 4y_p''(t) = & -4A\cos t & -4B\sin t \end{array}$$

Therefore,

$$4y_p''(t) - 3y_p'(t) - y(t) = (-5A - 3B)\cos t + (-5B + 3A)\sin t$$

and

$$-5A + 3B = -2, \quad -5B + 3A = 0,$$

$$B = \frac{3}{5}A, \quad -5A - 3\frac{3}{5}A, = \frac{-25-9}{5}A = -\frac{-34}{5}A = -2,$$

$$A = \frac{5}{17}, \quad B = \frac{3}{5}A = \frac{3}{5}\cdot\frac{5}{17} = \frac{3}{17}$$

Example 3. *For the differential equation*

$$3y'' - 2y' + 6y = e^{2t},$$

we look for a solution of the form $y_p(t) = Ae^{2t}$ with undetermined coefficient A. This suffices because the derivative of $y_p(t)$ are all multiples of e^{2t}. Then

$$3y'' - 2y' + 6y = 12Ae^{2t} - 4Ae^{2t} + 6Ae^{2t} = 14Ae^{2t}$$

Thus, $A = 1/14$ and

$$y_p(t) = \frac{1}{14}e^{2t}$$

is a particular solution.

Example 4. *For the differential equation*

$$y'' + y' - 6y = e^{2t},$$

we have the auxiliary equation

$$r^2 + r - 6 = (r-2)(r+3) = 0.$$

In this case, e^{2t} is a solution to the homogeneous equation. As in the case of double roots, we add a term te^{2t} and look for a particular solution of the form

$$y_p(t) = (At + B)e^{2t}.$$

$$\begin{array}{rll} y_p(t) = & Ate^{2t} & +Be^{2t} \\ y_p'(t) = & 2Ate^{2t} & +Ae^{2t} + 2Be^{2t} \\ y_p''(t) = & 4Ate^{2t} & +2Ae^{2t} + 2Ae^{2t} + 4Be^{2t} \end{array}$$

and

$$\begin{array}{rll} -6y_p(t) = & -6Ate^{2t} & -6Be^{2t} \\ y_p'(t) = & 2Ate^{2t} & +(A + 2B)e^{2t} \\ y_p''(t) = & 4Ate^{2t} & +(4A + 4B)e^{2t} \end{array}$$

$$y_p''(t) + y_p'(t) - 6y_p(t) = 5Ae^{2t}$$

and $A = 1/5$. Thus,

$$y_p(t) = \frac{1}{5}te^{2t}.$$

The inclusion of the B term was not necessary.

In general, when we have a nonhomogeneous differential equation

$$ay'' + by' + cy = p(t)e^{rt}$$

for $p(t)$ a polynomial of degree m, we take

$$y_p(t) = t^s q(t) r^{rt}$$

where q is a polynomial of degree m and $s = 0, 1, 2$ is the number of times r is a solution to the auxiliary equation.

2 Variation of Parameters

Variation of parameters, also known as **variation of constants**, is a more general method to solve inhomogeneous linear ordinary differential equations.

For first-order inhomogeneous linear differential equations, we were able to determine a solution using an integrating factor. For second order equations, we have used a heuristic approach they may fail for many choices of f.

2.1 Review of First Order Equations

To solve for a first order equation

$$y' + p(t)y = q(t) \tag{1}$$

Recall that the general solution of the corresponding homogeneous equation

$$y' + p(t)y = 0$$

This homogeneous differential equation can be solved by different methods, for example separation of variables, we note that is equation is separable.

$$\begin{aligned}
\frac{dy_h}{dt} + p(t)y_h &= 0 \\
\frac{dy_h}{dt} &= -p(t)y_h \\
\frac{1}{y_h}\frac{dy_h}{dt} &= -p(t)y_h \\
\int \frac{1}{y_h}\frac{dy_h}{dt} dt &= -\int p(t)\,dt \\
\int \frac{1}{y_h} dy_h &= -\int p(t)\,dt \\
\ln|y_h| &= -\int p(t)\,dt + c \\
|y_h| &= \exp(-\int p(t)\,dt + c) \\
y_h &= A\exp\left(-\int p(t)\,dt\right) = A\mu(t)
\end{aligned}$$

Now we return to solving the non-homogeneous equation (1). The method variation of parameters forms the particular solution by multiplying solution by an unknown function $v(t)$

$$y_p = v(t)\mu(t)$$

By substituting y_p into the non-homogeneous equation, (1) we can find v.

$$\begin{aligned}
y_p' + py_p &= q \\
(v\mu)' + p(v\mu) &= q \\
v''\mu + v\mu' + p(v\mu) &= q \\
v''\mu + v(\mu' + p\mu) &= q \\
v'\mu &= q \quad \text{(Note that } \mu' + p\mu = 0. \\
v' &= q/\mu \\
v(t) &= \int q(t)/\mu(t)dt \\
v(t) &= \int q(t)\exp\left(\int p(t)\,dt\right) dt
\end{aligned}$$

We only need a single particular solution, so we can set the constant of integration to be 0.

Now we have the solution

$$y(t) = y_h(t) + y_p(t).$$

2.2 General Procedure for Second Order Equation

Write

$$y_h(t) = c_1y_1(t) + c_2y_2(t)$$

as the general solution to

$$ay'' + by' + cy = 0. \tag{2}$$

Now, we seek a solution of

$$ay'' + by' + cy = f \tag{3}$$

of the form

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) \tag{4}$$

The derivative, by the product rule, is

$$y_p'(t) = (v_1'(t)y_1(t) + v_2'(t)y_2(t)) + (v_1(t)y_1'(t) + v_2(t)y_2'(t)).$$

In order to avoid second derivatives on v_1 and v_2, we attempt a solution with

$$0 = v_1'(t)y_1(t) + v_2'(t)y_2(t) \tag{5}$$

and

$$y_p'(t) = v_1(t)y_1'(t) + v_2(t)y_2'(t). \tag{6}$$

Take one more derivative.

$$y_p''(t) = v_1'(t)y_1'(t) + v_1(t)y_1''(t) + v_2'(t)y_2'(t) + v_2(t)y_2''(t). \tag{7}$$

We now substitute this into (3) and simplify,

$$\begin{aligned}
f &= ay_p'' + by_p' + cy_p \\
f &= a(v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'') + b(v_1y_1' + v_2y_2') + c(v_1y_1 + v_2y_2) \\
f &= a(v_1'y_1' + v_2'y_2') + v_1(ay_1' + by_1' + c) + v_2(ay_2' + by_2' + c) \\
f &= a(v_1'y_1' + v_2'y_2') \\
\frac{1}{a}f &= v_1'y_1' + v_2'y_2' \qquad (8)
\end{aligned}$$

Here, we use the fact that y_1 and y_2 solve the homogeneous equation (2).

Now we have two linear equations for v_1' and v_2', namely, (5) and (8). We can solve for v_2 in (5), substitute into (8), simplify and then substitute the solution of v_1 back into (5). This results in the solutions

$$v_1' = -\frac{fy_2}{a(y_1y_2' - y_1'y_2)} \quad \text{and} \quad v_2' = \frac{fy_2}{a(y_1y_2' - y_1'y_2)}.$$

Finally, we integrate,

$$v_1(t) = -\int \left(\frac{f(t)y_2(t)}{a(y_1(t)y_2'(t) - y_1(t)'y_2(t))}\right) dt \quad \text{and} \quad v_2(t) = -\int \left(\frac{f(t)y_2(t)}{a(y_1(t)y_2'(t) - y_1'(t)y_2(t))}\right) dt.$$

and substitute into (4).

Remark 5. *The expression $W(y_1, y_2) = y_1y_2' - y_1'y_2$ is the 2×2 example of the Wronskian Written as a determinant*

$$W(y_1, y_2) = det \begin{pmatrix} y_1 & y_2 \\ y_1] & y_2' \end{pmatrix}$$

Then

$$v_1' = -\frac{fy_2}{aW(y_1, y_2)} \quad \textit{and} \quad v_2' = \frac{fy_2}{aW(y_1, y_2)}.$$

2.3 Examples

Rather than using the general expression for the solution, we will follow the process to derive these equations beginning with

$$\begin{aligned} 0 &= v_1'(t)y_1(t) + v_2'(t)y_2(t) \\ \frac{1}{a}f(t) &= v_1'(t)y_1'(t) + v_2(t)'y_2'(t) \end{aligned}$$

to determine v_1 and v_2 and then substitute into

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

Example 6. *For*

$$y'' + 4y = t,$$

the auxiliary equation is

$$r^2 + 4 = 0.$$

with solutions $r_\pm = \pm 2i$. *This gives the solutions*

$$y_1(t) = \cos 2t \quad \textit{and} \quad y_2(t) = \sin 2t.$$

with derivatives

$$y_1'(t) = -2\sin 2t \quad \textit{and} \quad y_2'(t) = 2\cos 2t.$$

Thus,

$$\begin{aligned} 0 &= v_1'(t)\cos 2t + v_2'(t)\sin 2t \\ t &= -2v_1'(t)\sin 2t + 2v_2'(t)\cos 2t \end{aligned}$$

We solve for v_1', v_2', *integrate,*

$$v_2(t) = -\frac{\cos 2t}{\sin 2t} v_1(t)$$

$$\begin{aligned} t &= -2v_1'(t)\sin 2t - \frac{\cos 2t}{\sin 2t} v_1'(t)\cos 2t \\ t\sin 2t &= -2v_1'(t) \\ v_1'(t) &= -\frac{1}{2}t\sin t, \quad v_1(t) = \frac{1}{8}(2t\cos 2t - \sin 2t) \\ v_2'(t) &= \frac{\cos 2t}{\sin 2t} \cdot \frac{1}{2}t\sin 2t = \frac{1}{2}t\cos 2t \quad v_2(t) = \frac{1}{8}(2t\sin 2t + \cos 2t) \end{aligned}$$

and substitute into the equation for y_p

$$y_p(t) = \frac{1}{8}(2t\cos 2t - \sin 2t)\cos 2t + \frac{1}{8}(2t\sin 2t + \cos 2t)\sin 2t = \frac{1}{4}t.$$

Example 7. *To find a solution to*

$$y'' - 5y' + 6y = 2e^t$$

Note that the auxiliary equation for the homogeneous equation is

$$r^2 - 5r + 6 = (r-2)(r-3).$$

This gives two linearly independent solutions

$$y_1(t) = e^{2t} \quad \textit{and} \quad y = e^{3t}.$$

The Wronskian

$$W(y_1, y_2) = e^{2t}(3e^{3t}) - (2e^{2t})e^{3t} = -e^{5t}.$$

Thus,

$$v_1'(t) = -\frac{2e^t e^{2t}}{-e^{5t}} = 2e^{-2t} \quad \textit{and} \quad v_2'(t) = \frac{2e^t e^{3t}}{-e^{5t}} = -2e^{-t}.$$

$$v_1(t) = - = e^{-2t} \quad \textit{and} \quad v_2(t) = e^{-t}$$

and

$$y_p(t) = (-e^{-2t})(e^{3t}) + (2e^{-t})(e^{2t}) = e^t.$$

A general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t).$$