Introduction to the Phase Plane

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1 The Phase Line

A single first order differential equation of the form

$$\frac{dy}{dt} = f(y) \tag{1}$$

makes no mention of t in the function f.

Such a differential equation is called **autonomous**, that is time independent variable t does not appear explicitly. When the independent variable is time, then the system is also known as **time invariant**. This is the content of the next exercise.

Exercise 1. *If $y(t))$ is a solution of (2), and t_0 is a real number, then so is $y(t_0 + t)$.*

We analyze the qualitative properties of (**??**) using the *phase line* through an example.

Example 2. *Consider the autonomous differential equation*

$$\frac{dy}{dt} = -A(y - y_1)(y - y_2)(y - y_3)^2.$$

with $y_1 < y_2 < y_3$. We can give the qualitative behavior using the phase line. This is a line with arrow in the plane pointing to the right at y if $f(y)$ is positive and to the left if $f(y)$ is negative. Thus.

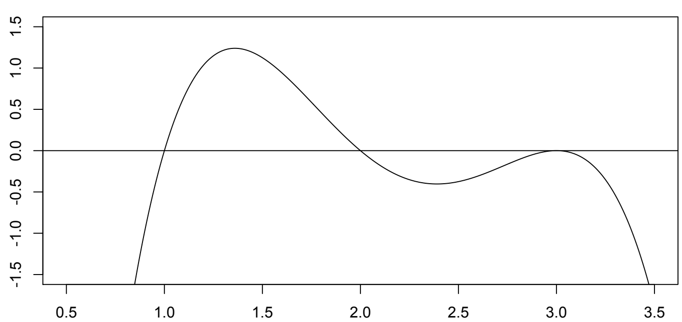

Figure 1: Plot of $-A(y - y_1)(y - y_2)(y - y_3)^2.$, $A = 2, y_1 = 1, y_2 = 2, y_3 = 3$

- *Solutions $y(t)$ move the direction of the arrows as time increases.*
- *A point y_0 are called an* **equilibrium** *of* **critical point** *if $f(y_0) = 0$. This give the trivial solution*

$$y(t) = y_0 \quad \textit{for all } t.$$

Example 3. *Describe the equilibrium points of*

$$\frac{dy}{dx} = \sin t.$$

For the equilibrium points

- *If both arrows point toward the critical point, it is called* **stable** *or a* **sink**. *Nearby solutions will converge to the equilibrium point. The solution is stable under small perturbations, meaning that if the solution is disturbed, it will return. y_2 is a stable point.*
- *If both arrows point away from the critical point, it is called* **unstable** *or* **source**. *Nearby solutions will diverge from the equilibrium point, The solution is unstable under small perturbations, meaning that if the solution is disturbed, it will not return. y_1 is an unstable point.*
- *If one arrow points towards the equilibrium point, and one points away, it is* **semi-stable** *or a* **node***: It is stable in one direction (where the arrow points towards the point), and unstable in the other direction (where the arrow points away from the point). y_2 is a node.*

More generally, if f has a derivative.

- $f'(y) < 0$ at stable equilibria,
- $f'(y) > 0$ at unstable equilibria.
- $f'(y) = 0$ at nodes.

2 Representation as a System of First Order Equations

For the second order linear ordinary differential equation.

$$y'' + by' + cy = f.$$

We can write this as system of two first order equations

$$y' = v \quad \text{and} \quad v' = (bv + cy)/a.$$

In the case of mass-spring oscillator, v is the velocity. In the case of a circuit in series, we typically write the equation in terms of the charge q,

$$aq'' + bq' + cq = E,$$

we can write this as the system

$$q' = I \quad \text{and} \quad I' = (bI + cq)/a.$$

In this case I is the current.

3 The Phase Plane Equation

Now we will look at a system of two first order linear ordinary differential equations.

$$\frac{dx}{dt} = f(x, y) \quad \text{and} \quad \frac{dy}{dt} = g(x, y) \tag{2}$$

This system is also called **autonomous** and **time invariant**. As above, we have the property.

Exercise 4. *If $(x(t), y(t))$ is a solution of (2), and t_0 is a real number, then so is $(x(t_0 + t), y(t_0 + t))$.*

By the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

or

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

For time invariant systems in (2), this allows us to consider the **phase plane equation**,

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)} \tag{3}$$

Let us review what the phase plane equation tells us and what it does not tell us.

- A plot of $(x(t), y(t))$ is a solution of (2) is a curve in the plane, known as a **trajectory**.
- If $(x(t), y(t))$ is a solution of (2), then the derivatives of the solution are vectors

  $$(dx/dt, dy/dt) = (f(x, y), g(x, y))$$

  with

  - slope $dy/dx = g(x, y)/f(x, y)$ and
  - magnitude (or speed)

  $$\sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{f(x, y)^2 + g(x, y)^2}$$

- The trajectory itself does not tell us when the solution arrives at a position (x, y) in the plane, not does it tell us the speed. Indeed, if we were t0 multliply both f and g by a constant $\alpha \neq 0$, it would not change the trajectory. It would change the speed and it would change the direction of the solution curve is $\alpha < 0$.
- It does tell us the slope dy/dx which is the tangent line to the trajectory at a position (x, y).
- The solutions to (3) give us these trajectories.
- However, for (2), t is the independent variable. x and y are dependent variables.
- The direction fields give us solutions to differential equations with x the independent variable and y dependent. So, even though these plots look very similar, they reveal different information.

Example 5. *An LC circuit, also called a resonant circuit,* **tank circuit***, or* **tuned circuit***, is an electric circuit consisting of an inductor, having inductance L, and a capacitor, having capacitance C, the governing equations for the chage q are*

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0.$$

If we have the inttial value problem, $q(0) = 0$, $q'(0) = I_0$, then the solution is

$$q(t) = \frac{I_0}{\omega_0}\sin\omega_0 t$$

where $\omega_0^2 = 1/LC$. The current,

$$I(t) = \frac{dq}{dt}(t) = I_0\cos\omega_0 t.$$

Thus the trajectory

$$(q(t), I(t)) = I_0(\frac{1}{\omega_0}\sin\omega_0 t, \cos\omega_0 t).$$

is an ellipse.

$$\omega_0^2 q^2 + I^2 = I_0.$$

In addition,

$$\frac{dI}{dt}(t) = \frac{d^2q}{dt^2}(t) = -I_0\omega_0\sin\omega_0 t.$$

In the qIplane the solution $(q(t), I(t))$ has slope

$$\frac{dI}{dq} = \frac{dI/dt}{dq/dt} = \frac{-I_0\omega_0\sin\omega_0 t}{I_0\cos\omega_0 t.} = -\omega_0\tan\omega_0 t.$$

If now, we look at this same equation at a first order system, we have that

$$\frac{dq}{dt} = I, \quad \textit{and} \quad \frac{dI}{dt} = -\frac{1}{LC}q$$

Thus,

$$\frac{dI}{dq} = \frac{dI/dt}{dq/dt} = -\frac{q/LC}{I} = -\frac{\omega_0^2 q}{I}$$

From the solution to the differential equation, we have

$$-\frac{I}{\omega_0^2 q} = -\frac{I_0\sin\omega_0 t}{\omega_0 I_0\cos\omega_0 t} = -\frac{1}{\omega_0}\cot\omega_0 t$$

as before.

If we focus on a solution to

$$\begin{aligned}
\frac{dI}{dq} &= -\frac{\omega_0^2 q}{I} \\
I\frac{dI}{dq} &= -\omega_0^2 q \\
\int I\frac{dI}{dq}dq &= -\int \omega_0^2 q dq \\
\frac{1}{2}I^2 &= -\frac{1}{2}\omega_0^2 q^2 + c \\
\omega_0^2 q^2 + I^2 &= 2c
\end{aligned}$$

If the trajectory has the point $(0, I_0)$, then $2c = I_0^2$, the ellipse we saw before. The difference is that we do not have the time course for the trace of the differential equation along the ellipse.

4 Equilibrium Points

As in the of one equation, a point x_0, y_0 satisfying

$$f(x_0, y_0) = 0 \quad \text{and} \quad g(x_0, y_0) = 0$$

called a **critical point**, or **equilibrium point**, of the autonomous system (2)

The corresponding constant solution $(x(t), y(t)) = (x_0, y_0)$ is called an **equilibrium solution**. The set of all critical points is called the **critical point set**.

Example 6. *For*

$$\frac{dx}{dt} = -x, \quad \textit{and} \quad \frac{dy}{dt} = -2y$$

The direction field

$$\frac{dy}{dx} = \frac{2y}{x}$$

Gives vectors with positive slope in quadrants I and III and negative slope in quadrants I and IV. So the vector etiher aim toward the equilibrium point $(0, 0)$ or away. Since both x and y are decreasing with time in quadrant I, we can add arrows that point to the origin and the origin is stable.

To solve the phase plane equation

$$\begin{aligned}
\frac{1}{2y}\frac{dy}{dx} &= \frac{1}{x} \\
\int \frac{1}{2y}\frac{dy}{dx}dx &= \int 1x dx \\
\int \frac{1}{2y}dy &= \int 1x dx \\
\frac{1}{2}\ln|y| &= \ln|x| + c \\
y &= cx^2
\end{aligned}$$

For

$$\frac{dx}{dt} = x, \quad \textit{and} \quad \frac{dy}{dt} = 2y$$

We have the same equilibrium point and the same phase plane equation, but the origins is unstable and the arrows point autward

Exercise 7. *Give the equilibrium points and use software to describe them for the following pairs of differential equations (See* `http://comp.uark.edu/~aeb019/pplane.html`*.)*

1. $$\frac{dx}{dt} = y - \frac{x}{} \textit{and} \quad \frac{dy}{dt} = \sin x$$

2. $$\frac{dx}{dt} = x + y \quad \textit{and} \quad \frac{dy}{dt} = 1 - x^2$$

3. $$\frac{dx}{dt} = y \quad \textit{and} \quad \frac{dy}{dt} = x - x^3$$

We can now develop a general procedure:

If (x_0, y_0) is an equilibrium point.

- Create a linearize equation about this point, setting $(\tilde{x}, \tilde{y}) = (x - x_0, y - y_0)$

$$\frac{d\tilde{x}}{dt} = \tilde{x}\frac{\partial f}{\partial x}(x_0, y_0) + \tilde{y}\frac{\partial f}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{dy}{dt} = \tilde{x}\frac{\partial g}{\partial x}(x_0, y_0) + \tilde{y}\frac{\partial g}{\partial y}(x_0, y_0)$$

- Create a single second order linear constant coefficient ordinary differential equation.
- Find the roots r_- and r_+ of the associated auxiliary equation.
- The type of equilibrium point is determined by the following table

r_- and r_+	type of equilibrium point
real and both negative	stable
real and both positive	unstable
real, one positive and one negative	saddle node
purely imaginary	oscillatory
complex, negative real part	stable spiral
complex, positive real part	unstable spiral