Likelihood Ratio Tests

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The likelihood ratio test is a popular choice a composite hypothesis.

\[ H_0 : \theta \in \Theta_0 \] versus \[ H_1 : \theta \in \Theta_1 \]

when \( \Theta \) is a multidimensional parameter space and \( \Theta_0 \) is a subspace.

\[ \Lambda(x) = \frac{\sup \{ L(\theta|x) ; \theta \in \Theta_0 \}}{\sup \{ L(\theta|x) ; \theta \in \Theta \}} \]

The rejection region for an \( \alpha \)-level test is \( \{ \Lambda(x) \leq \lambda_0 \} \) where \( \lambda_0 \) is chosen so that

\[ P_\theta \{ \Lambda(X) \leq \lambda_0 \} \leq \alpha \]

for all \( \theta \in \Theta_0 \).

Let \( \hat{\theta}_0 \) be the parameter value that maximizes the likelihood for \( \theta_0 \in \Theta_0 \) and \( \hat{\theta} \) be the parameter value that maximizes the likelihood for \( \theta_0 \in \Theta \). Then,

\[ \Lambda(x) = \frac{L(\hat{\theta}_0|x)}{L(\hat{\theta}|x)}. \]

**Example 1.** Let \( \Theta = \mathbb{R} \) and consider the two-sided hypothesis

\[ H_0 : \mu = \mu_0 \] versus \[ H_1 : \mu \neq \mu_0. \]

Here the data are \( n \) independent \( N(\mu, \sigma^2) \) random variables \( X_1, \ldots, X_n \) with known variance \( \sigma^2 \). Then, \( \hat{\mu}_0 = \mu_0 \) and \( \bar{\mu} = \bar{x} \). Consequently,

\[ L(\hat{\mu}_0|x) = \left( \frac{1}{2\pi\sigma^2} \right)^n \exp - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_0)^2, \quad L(\hat{\mu}|x) = \left( \frac{1}{2\pi\sigma^2} \right)^n \exp - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \bar{x})^2 \]

and

\[ \Lambda(x) = \exp - \frac{1}{2\sigma^2} \left( \sum_{i=1}^{n} ((x_i - \mu_0)^2 - (x_i - \bar{x})^2) \right) = \exp - \frac{n}{2\sigma^2} (\bar{x} - \mu_0)^2. \]

Now notice that

\[ -2 \ln \Lambda(x) = \frac{n}{\sigma^2} (\bar{x} - \mu_0)^2 = \left( \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \right)^2. \]

Because \( (\bar{X} - \mu_0)/(\sigma/\sqrt{n}) \) is a standard normal random variable, \( -2 \ln \Lambda(X) \) is the square of a standard normal, hence, a \( \chi^2 \)-square random variable with 1 degree of freedom.
1 Chi-square test

This exact computation for normal data yields, owing to the central limit theorem, an asymptotic result that is contained in the following theorem.

**Theorem 2.** Whenever the maximum likelihood estimate has an asymptotically normal distribution, let $\Lambda_n(x)$ be the likelihood ratio criterion for $H_0: \theta_1 = c_1$ for all $i = 1, \ldots, k$ versus $H_1: \theta_1 \neq c_1$ for some $i = 1, \ldots, n$

Then under $H_0$,

$$-2 \ln \Lambda_n(X)$$

converges in distribution to a $\chi^2_k$ random variable.

**Example 3.** Let $X_1 \ldots X_n$ be independent $\text{Pois}(\lambda)$ random variables and consider the test

$$H_0: \lambda = \lambda_0 \quad \text{versus} \quad H_1: \lambda \neq \lambda_0.$$

Then the likelihood

$$L_n(\theta|x) = \frac{\lambda^{x_1}}{x_1!} e^{-\lambda} \cdots \frac{\lambda^{x_n}}{x_n!} e^{-\lambda} = \frac{1}{x_1! \cdots x_n!} \lambda^{x_1+\cdots+x_n} e^{-n\lambda}$$

The maximum likelihood is taken for $\lambda = \bar{x}$. Thus,

$$\Lambda(x) = \frac{\lambda_0^{x_1+\cdots+x_n} e^{-n\lambda_0}}{\bar{x}^{x_1+\cdots+x_n} e^{-n\bar{x}}} = \frac{\bar{x}^{n\bar{x}} e^{-n\bar{x}}}{\lambda_0^{n\lambda_0} e^{-n\lambda_0}}$$

and

$$-2 \ln \Lambda_n(X) = -2n(\bar{x} \ln \lambda_0 - \lambda_0 - \bar{x} \ln \bar{x} + \bar{x}).$$

To determine the critical values for this test, we have

```r
> qchisq(c(0.90,0.95,0.98,0.99),1)
[1] 2.705543 3.841459 5.411894 6.634897
```

We compare $-2 \ln \Lambda_n(X)$ to the $\chi^2_1$ distribution with $n = 36$ and $\lambda = 3$ using 1000 simulations under the null hypothesis $\lambda = 3$.

```r
> neg2lnlambda <-rep(0,1000)
> n=36
> lambda=3
> for(i in 1:1000){x<-rpois(n,3);
 neg2lnlambda[i] =-2*n*(mean(x)*log(lambda)-lambda-mean(x)*log(mean(x))+mean(x))}
> hist(neg2lnlambda,probability=TRUE)

and

> curve(dchisq(x,1),0,12)
```
Histogram of neg2lnlambda

Density

0 2 4 6 8 10 12
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7

x
dchisq(x, 1)