Applications

June 23, 2016

1 Examples from Biology

1.1 Population Models

The simplest model, introduced by Thomas Malthus, are for populations unconstrained in their growth witth this growth is proportional to its size. The parameter α is an intrinsic growth rate.

$$\frac{dp}{dt} = \alpha p$$

with solution

$$p(t) = p_0 e^{\alpha t}$$

In his 1798 boook, *An Essay on the Principle of Population,* Malthus wrote

> *That the increase of population is necessarily limited by the means of subsistence,*
> *That population does invariably increase when the means of subsistence increase,*
> *and,*
> *That the superior power of population is repressed, and the actual population kept equal to the means of subsistence, by misery and vice.*

If the population is subject to a constraint in size, say its maximum size is M, then we have a model

$$\frac{dp}{dt} = \alpha p \left(1 - \frac{p}{M}\right) \tag{1}$$

The parameter M is called the carrying capacity.

Exercise 1. *Give a phase line analysis of the equilibrium points of (1).*

Separation of variables give a solution

$$\begin{aligned}
\ln\left(\frac{p}{M-p}\right) &= \alpha t + c \\
\frac{p}{M-p} &= Ae^{\alpha t} \\
p &= \frac{MAe^{\alpha t}}{1 + Ae^{\alpha t}} \\
p &= \frac{MA}{Ae^{-\alpha t} + 1}
\end{aligned}$$

If the death rate is proportional to the population size, then the model becomes with death rate $\gamma < \alpha$,

$$\begin{aligned}
\frac{dp}{dt} &= \alpha p\left(1 - \frac{p}{M}\right) - \gamma p \\
&= p\left(\alpha - \frac{\alpha p}{M} - \gamma\right) \\
&= p\left(\alpha - \gamma - \frac{\alpha p}{M}\right) \\
&= (\alpha - \gamma)p\left(1 - \frac{\alpha p}{(\alpha - \gamma)M}\right)
\end{aligned}$$

This the model has the same structure with lower intrinsic growth rate $k - \gamma$ and reduced carrying capacity $(k - \gamma)M/k$

Next we consider a two species model.

The **Lotka-Volterra equations**, also known as the **predator-prey equations**, is frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey. The populations change through time according to the pair of equations:

$$\frac{dx}{dt} = \alpha x - \beta xy \quad \text{and} \quad \frac{dy}{dt} = \delta xy - \gamma y \tag{2}$$

where

- x is the number of prey (for example, rabbits)
- y is the number of some predator (for example, foxes); and
- α, β, γ, and δ are positive real parameters describing the interaction of the two species.

Thus,

- The prey are assumed to have an unlimited food supply, and to be increasing at growth rate α
- The prey is subject to predation at a rate that is proportional its population size and to the population size of the predator. The constant of proportionality is β
- The predator population increases at a rate proportional to the amount of resources consumes. This, too, is proportional its population size and to the population size of the prey. The constant of proportionality is δ
- The predator death rate is γ

The critical point with $x_0 > 0$ and $y_0 > 0$.

$$(x_0, y_0) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$$

The phase plane equation has an implicit solution, found using separation of variables.

$$\begin{aligned}
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} &= \frac{\delta xy - \gamma y}{\alpha x - \beta xy} = \frac{y(\delta x - \gamma)}{x(\alpha - \beta y)} \\
\frac{\alpha - \beta y}{y}\frac{dy}{dx} &= \frac{\delta x - \gamma}{x} \\
\int \frac{\alpha - \beta y}{y}\frac{dy}{dx}dx &= \int \frac{\delta x - \gamma}{x}dx \\
\int \frac{\alpha - \beta y}{y}dy &= \int \frac{\delta x - \gamma}{x}dx \\
(\alpha \ln y - \beta y) &= (-\gamma \ln x + \delta x) + c
\end{aligned}$$

1.2 Epidemic Models

The **SIR model** divides a population into three groups from a population of size N

- S is the number susceptible,
- I this number infectious, and
- R is the number recovered (immune or removed).

This is a good and simple model for many infectious diseases including measles, mumps and rubella. If the dynamics of an epidemic, for example the flu, are often much faster than the dynamics of human birth , then, birth is often omitted.

The differential equation model

$$\begin{aligned}
\frac{dS}{dt} &= -\frac{\beta IS}{N} \\
\frac{dI}{dt} &= \frac{\beta IS}{N} - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{aligned}$$

- Susceptible individuals become infected at a rate proportional to the number of encounters with infected individuals, i. e., proportional to both the susceptible population size and the infected population size. The constant of proportionality is β
- Infected individuals become recovered at rate proportional to the infected population. These individuals become recovered (or immune or removed). The constant of proportionality is γ
- Because
$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0.$$
the sum
$$S(t) + I(t) + R(t)$$
is constant. The value of this constant is N.

Thus, the model does not require the final differential equation. $R(t) = N - S(t) - I(t)$. This is a version of the Lotka-Volterra equation with $S = x$, $I = y$, $\alpha = 0$, and $\delta = \beta$ in (2).

The phase plane equation,

$$\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \frac{\beta IS/N - \gamma I}{-\beta IS/N} = -1 + \frac{\gamma N}{\beta S}$$

which has solution

$$I = -S + \frac{\gamma N}{\beta} \ln S + c.$$

2 Coupled Mass-Spring Systems

On a smooth horizontal surface, a mass $m_1 = 2$ kg is attached to a fixed wall by a spring with spring constant $k_1 = 4$ N/m. Another mass $m_2 = 1$ kg is attached to the first object by a spring with spring constant $k_2 = 2$ N/m. The objects are aligned horizontally so that the springs are their natural lengths Both objects are displaced 3 m to the right of their equilibrium positions and then released.

So the frequencies of the springs when they are not coupled are

$$\omega_1 = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{4}{2}} = \sqrt{2} \quad \text{and} \quad \omega_2 = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{2}{1}} = \sqrt{2}.$$

Using Hooke's law we have two forces on the inner mass

$$F_1 = -k_1 x \quad \text{and} \quad F_2 = k_2(y - x)$$

and one force on the outer mass

$$F_3 = -k_2(y - x).$$

By Newton's second law

$$\begin{aligned} m_1 \frac{d^2x}{dt^2} &= F_1 + F_2 = -k_1 x + k_2(y - x) \\ m_2 \frac{d^2y}{dt^2} &= F_3 = -k_2(y - x) \end{aligned}$$

or, rearranging

$$\begin{aligned} m_1 \frac{d^2x}{dt^2} + (k_1 + k_2)x - k_2 y &= 0 \\ m_2 \frac{d^2y}{dt^2} + k_2 y - k_2 x &= 0 \end{aligned}$$

We analyze a specific example with $m_1 = 2$, $m_2 = 1$, $k_1 = 4$,and $k_2 = 2$. Then

$$\begin{aligned} 2\frac{d^2x}{dt^2} + 6x - 2y &= 0 \\ \frac{d^2y}{dt^2} + 2y - 2x &= 0 \end{aligned}$$

The goal it to find a single equation in the one of the variables. Choosing x, we take two derivatives and eliminate the d^2u/dtr term. This introduces a term in y we can remove using the first equation above.

$$\begin{array}{cccccc} 2\frac{d^4x}{dt^4} & +6\frac{d^2x}{dt^2} & -2\frac{d^2y}{dt^2} & & & =0 \\ & & +2\frac{d^2y}{dt^2} & -4x & +4y & =0 \\ & +4\frac{d^2x}{dt^2} & & +12x & -4y & =0 \\ \hline 2\frac{d^4x}{dt^4} & +10\frac{d^2x}{dt^2} & & +8x & & =0 \end{array}$$

Thus, we look to solve a fourth order, linear, constant coefficient ordinary differential equation. If we substitute e^{rt} into this equation, we have the auxiliary equation that is a polynomial of degree 4.

$$2r^4 + 10r^2 + 8 = 2(r^4 + 5r^2 + 4) = 2(r^2 + 1)(r^2 + 4) = 0.$$

The roots are $\pm i, \pm 2i$, yielding 4 linearly independent solutions

$$x_1(t) = \cos t \quad , x_2(t) = \sin t, \quad , x_3(t) = \cos 2t, \quad x_4(t) = \sin 2t$$

and the general solution

$$x(t) = a_1 \cos t + a_2 \sin t + a_3 \cos 2t + a_4 \sin 2.t$$

We return to the governing equations to see that

$$\begin{aligned} y(t) = \frac{d^2x}{dt^2} + 3x &= -(a_1 \cos t + a_2 \sin t + 4a_3 \cos 2t + 4a_4 \sin 2t) + 3(a_1 \cos t + a_2 \sin t + a_3 \cos 2t + a_4 \sin 2t) \\ &= 2a_1 \cos t + 2a_2 \sin t - a_3 \cos 2t - a_4 \sin 2t \end{aligned}$$

To match the initial conditions,

$$x(0) = 3, \quad \frac{dx}{dt} = 0, \quad y(0) = 3, \quad \frac{dy}{dt} = 0.$$

we substitute into the equations for x and y.

$$3 = x(0) = a_1 + a_3, \quad 3 = y(0) = 2a_1 - a_3 \quad \text{Thus,} \quad a_1 = 2, \quad a_3 = 1.$$

to determine a_2 and a_4, we take derivatives,

$$\begin{aligned} \frac{dx}{dt}(t) &= -a_1 \sin t + a_2 \cos t - 2a_3 \sin 2t + 2a_4 \cos 2t \\ \frac{dy}{dt}(t) &= -2a_1 \sin t + 2a_2 \cos t + 2a_3 \sin 2t - 2a_4 \cos 2t \end{aligned}$$

$$0 = \frac{dx}{dt}(0) = a_2 + 2a_4 \quad 0 = \frac{dy}{dt}(0) = 2a_2 - 2a_4. \quad \text{Thus,} \quad a_2 = 0, \quad a_4 = 0.$$

and

$$x(t) \;=\; 2\cos t + 2\cos 2t, \quad y(t) = 4\cos t - \cos 2t$$

Plot of the motion of the two springs is shown. Here is the R code

```
> t<-seq(0,8*pi,0.01)
> plot(t,2*cos(t)-2*cos(2*t)+3,type="l",ylim=c(-5,6),col="red",xlab="",ylab="")
> par(new=TRUE)
> plot(t,4*cos(t)-cos(2*t),type="l",ylim=c(-5,6),col="blue",xlab="",ylab="")
```

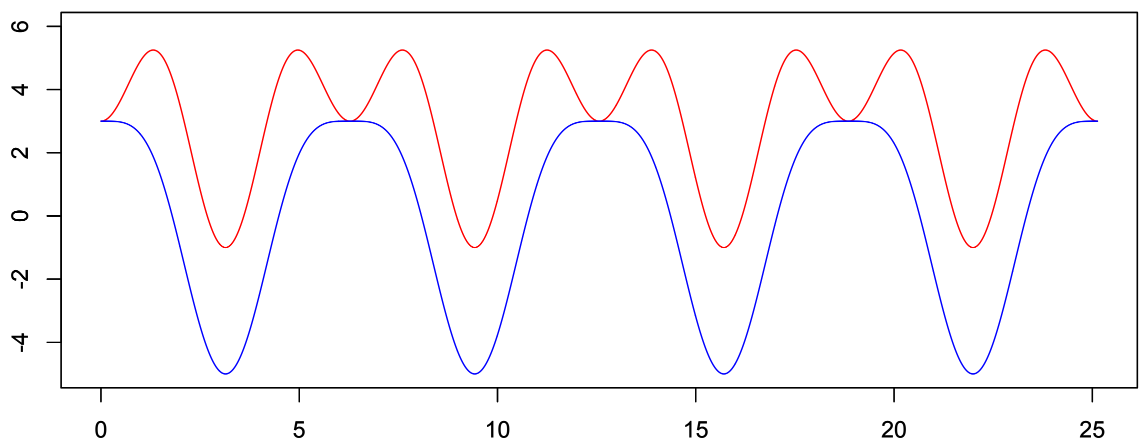

Figure 1: Plot of the movement of coupled springs, the displacement of the inner spring x is in red and of the outer spring y is in blue.