Polynomial Approximations and Power Series

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1 Tangent Lines

One of the first uses of the derivatives is the determination of the tangent as a linear approximation of a differentiable function f. By the definition of the deriviative

$$f'(x_0) \approx \frac{f(x) - f(x_0)}{x - x_0}$$

and thus

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) = p_1(x).$$

Exercise 1. $p_1(x_0) = f(x_0)$ *and* $p_1'(x_0) = f'(x_0)$

2 Taylor Polynomials

The Taylor polynomial of degree 2

$$p_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2$$

agrees at x_0 for f, f', and $f''(x)$. Taking derivatives,

$$\begin{array}{rlrl} p_2(x) = & a_0 + a_1(x - x_0) + a_2(x - x_0)^2 & p_2(x_0) & = a_0 \\ p_2'(x) = & a_1 + 2a_2(x - x_0) & p_2(x_0) & = a_1 \\ p_2''(x) = & 2a_2 & p_2(x_0) & = 2a_2 \end{array}$$

Thus,

$$a_0 = f(x_0), \quad a_1 = f'(x_0) \quad a_2 = \frac{1}{2} f''(x_0).$$

Exercise 2. *Find the second order Taylor polynomial for* $\sin x$, e^x, $\ln(x + 1)$, *and* $\sqrt{x + 1}$ *at* $x_0 = 0$

If we continue, asking for a polynomial

$$p_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n = \sum_{j=0}^{n} a_j (x - x_))^j$$

of degree n so that p_n and its first n derivatives agree with the corresponding values for f and its iirst n derivatives. Then, for the third derivative

$$f'''(x_0) = p_n'''(x_0) = 3 \cdot 2 \cdot 1 a_3 \quad a_3 = \frac{1}{3 \cdot 2 \cdot 1} f'''(x_0) = \frac{1}{3!} f'''(x_0),$$

and the fourth derivative

$$f'^{(4)}(x_0) = p_n^{(4)}(x_0) = 4 \cdot 3 \cdot 2 \cdot 1 a_4 \quad a_4 = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} f'''(x_0) = \frac{1}{4!} f(4)(x_0),$$

and, finally, for

$$f'^{(n)}(x_0) = p_n^{(n)}(x_0) = n! a_n \quad a_n = \frac{1}{n!} f^{(n)}(x_0),$$

Notice that the $n+1$-st derivative of $p_n(x)$ is 0. Also, note that the value

$$a_k = \frac{1}{k!} f^{(k)}(x_0)$$

does not depend on n as long as $k \leq n$.

Exercise 3. *Find the fourth order Taylor polynomial for* $\sin x$, $\ln x$, *and* e^x *at* $x_0 = 1$

Exercise 4. *Find the seventh order Taylor polynomial for* $\sin x$ *and* e^x *at* $x_0 = 0$

Exercise 5. *Approximate* $\sin \pi/3$ *using its seventh order Taylor polynomial.*

We do not need to know a function explicitly to compute its Taylor polynomial. For example, if

$$y' = x - 2y, \quad y(0) = 1. \tag{1}$$

Then,

$$\begin{aligned} y' &= x - 2y, & y'(0) &= 0 - 2y(0) = -2 \\ y'' &= 1 - 2y' & y''(0) = 1 - 2y'(0) &= 1 - 2(-2) = 5 \\ y''' &= -2y'' & y'''(0) = -2y''(0) &= -10 \\ y'''' &= -2y''' & y''''(0) = -2y'''(0) &= 20 \end{aligned}$$

and the four order polynomial for y at $x_0 = 0$ is

$$1 + -2x - \frac{5}{2}x^2 - \frac{5}{3}x^3 + \frac{5}{6}x^4.$$

Based on the generalized mean value theorem, careful analysis of the Taylor polynomial shows that we can obtain a bound on the error $E_n(x)$ between of the difference between $p_n(x)$ and $f(x)$, namely,

$$|E_n(x)| = |f(x) - p_n(x)| \leq \frac{M}{(n+1)!} |x - x_0|^{n+1},$$

where $M = \max f^{(n+1)}(x)$ on the interval between x_0 and x.

Exercise 6. *Find the solution to (1) and then find the fourth order polynomial for* y *at* $x_0 = 0$ *using this solution.*

Exercise 7. *Give an estimate on the error term for* $p_n(x)$ *for* $f(x) = \sin x$.

3 Power Series

The next question is: Can we continue this approximation indefinitely? Is

$$\lim_{n\to\infty}\sum_{j=0}^{n}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j=\sum_{j=0}^{\infty}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j fy(x)?$$

If so, for what values of x does the limit converge.

First, we have to investigate the question: When does the limit:

$$\lim_{n\to\infty}\sum_{j=0}^{n}a_j(x-x_))^j=\sum_{j=0}^{\infty}a_j(x-x_0)^j \tag{2}$$

exist? Again, if so, for what values of x does the limit **converge**.

We say that a power series (2) **converges absolutely** if

$$\sum_{j=0}^{\infty}|a_j(x-x_0)^j|$$

converges. If the limit fails to exist, we say that the power series **diverges**.

3.1 Radius of Convergence

A power series will converge for some values of the variable x and may diverge for others. All power series converge at $x = x_0$. There is always a number ρ with $0 \leq \rho \leq \infty$ such that the series converges absolutely whenever $|x-x_0| < \rho$ and diverges whenever $|x-x_0| > \rho$. This number is called the **radius of convergence**. The interval $(x_0-\rho, x_0+\rho)$ is called the **interval of convergence**.

The series may or may not converge for $|x-x_0| = \rho$. If (2) converges only for $x = x_0$, then $\rho = 0$. If it converges for all x, then we say that $\rho = \infty$.

Exercise 8. *The geometric series*

$$\sum_{j=0}^{\infty}ax^j$$

has radius of convergence $\rho = 1$. In this case, show that the infinite sum is

$$\frac{a}{1-x}.$$

What happens for $|x| = 1$?

The simplest test for a comparison test

If (2) converges absolutely for $x \in (x_0-\rho, x_0+\rho)$ and if for some number J, $|b_j| < |a_j|$ for all $j > J$, then

$$\sum_{j=0}^{\infty}b_j(x-x_0)^j$$

We have two tests for convergence based on comparisons with the geometric.

Ratio Test

If the limit

$$L = \lim_{j\to\infty} \left| \frac{a_j + 1}{a_j} \right|$$

exists, then in (2), the radius of convergence of (2) $\rho = L$.

Root Test

If the limit

$$\ell = \lim_{j\to\infty} \sqrt[j]{|a_j|}$$

exists, then in (2), the radius of convergence of $\rho = 1/\ell$. (Here, $1/0 = \infty$.)

Exercise 9. *Find the radius of convergence for*

$$\sum_{j=0}^{\infty} j^2 x^j, \quad \sum_{j=0}^{\infty} \frac{2^j}{j}(x-1)^j, \quad \textit{and} \quad \sum_{j=0}^{\infty} \frac{1}{j!} x^j$$

3.2 Arithmetic Operations

Let f and g be two power series converging in a interval about x_0.

$$\begin{aligned} f(x) &= \sum_{j=0}^{\infty} a_j (x - x_0)^j \\ g(x) &= \sum_{j=0}^{\infty} b_j (x - x_0)^j \end{aligned}$$

Addition and Subtraction

The power series of the sum or difference of the functions can be obtained by termwise addition and subtraction.

$$f(x) \pm g(x) = \sum_{j=0}^{\infty} (a_j \pm b_j)(x - x_))^j.$$

Multiplication

The power series for the product is sometimes called the **Cauchy product**.

$$f(x)g(x) = \sum_{j=0}^{\infty} c_j (x - x_0)^j$$

where

$$c_j = \sum_{i=0}^{j} a_i b_{j-i} = (a * b)_j.$$

This is is known as the **convolution** of the sequences a_j and b_j.

3.3 Uniqueness of Power Series

We begin with the following fact, which is somewhat technical to prove.

If a power series is equal to 0 is some interval then each of the coefficients a_j is 0. In particular, if two power series

$$\sum_{j=0}^{\infty} b_j(x-x_0)^j \quad \text{and} \quad \sum_{j=0}^{\infty} c_j(x-x_0)^j$$

agree for x in some interval, then the difference

$$0 = \sum_{j=0}^{\infty} b_j(x-x_0)^j - \sum_{j=0}^{\infty} c_j(x-x_0)^j = \sum_{j=0}^{\infty} (b_j - c_j)(x-x_0)^j$$

and so $b_j = c_j$ for all all j.

3.4 Differentiation and Integration

Power series are differentiable on the domain of convergence. So, for $x \in (x_0 - \rho, x_0 + \rho)$

Differentiation
Termwise differentiation gives the power series for the derivative.

$$f'(x) = \sum_{j=1}^{\infty} j a_j (x-x_0)^{j-1}$$

Because

$$\left|\frac{a_{j+1}}{a_j}\right| \to \rho, \quad \text{we have} \quad \left|\frac{(j+1)a_{j+1}}{j a_j}\right| \to \rho$$

and f' has the same radius of convergence as f.

Integration
Termwise integration gives the power series for the integral.

$$\int f(x)\,dx = \sum_{j=0}^{\infty} \frac{a_j}{j+1}(x-x_0)^{j+1} + c$$

$$\left|\frac{a_{j+1}}{a_j}\right| \to \rho, \quad \text{we have} \quad \left|\frac{a_{j+1}/(j+2}{a_j/(j+1)}\right| \to \rho$$

and $\int f$ has the same radius of convergence as f.

Exercise 10. *Use the fact that the geometric series*

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}.$$

to determine the power series for

$$\frac{1}{(1-x)^2} \quad \frac{1}{1+x} \quad \ln(1+x) \quad \frac{1}{1+x^2} \quad \tan^{-1}(x).$$

Exercise 11. *Show that*

$$\sum_{j=j_0}^{\infty} a_j x^j = \sum_{j=j_0+\ell}^{\infty} a_{j-\ell} x^{j-\ell}$$

3.5 Analytic Functions

A function f is called **analytic** at x_0 is f can be represented as a power series centered at x_0 with a positive radius of convergence.

$$f(x) = \sum_{j=0}^{\infty} a_n (x - x_0)^j, \quad |x - x_0| < \rho$$

Properties of Analytic Functions

- Polynomials are analytic for every value x_0.
- If f and g are analytic at x_0 so are $f + g$, fg, and f/g provided that $g(x_0) \neq 0$.
- All the derivatives of f are analytic with the same interval of convergence as f.
- The antiderivative of f is analytic with the same interval of convergence as f.
- The terms

  $$a_j = \frac{f^{(j)}(x_0)}{j!}$$

  Thus, the power series is the limit of the Taylor polynomials.
- Any power seriesregardless of how it is derived? that converges in some neighborhood of x_0 to a function is the Taylor series of that function.

Exercise 12. *Explain why*

$$(x - x_0)^p$$

is not analytic at x_0 *if* $p < 0$ *or if* $p > 0$ *and not an integer.*