Covariance and Correlation

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Here, we shall assume that the random variables under consideration have positive and finite variance.

One simple way to assess the relationship between two random variables $X$ and $Y$ is to compute their covariance.

$$\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)].$$

**Exercise 1.** Cov($aX + b, cY + d$) = $ac$ Cov($X, Y$).

**Example 2.** For the joint density example,

$$EXY = \frac{4}{5} \int_0^1 \int_0^1 xy(x + y + xy) dy dx = \frac{4}{5} \int_0^1 \int_0^1 (x^2y + xy^2 + x^2y^2) dy dx$$

$$= \frac{4}{5} \int_0^1 \left( \frac{1}{2}x^2y^2 + \frac{1}{3}xy^3 + \frac{1}{3}x^2y^3 \right) |_{y=0}^{y=1} dx = \frac{4}{5} \int_0^1 \left( \frac{5}{6}x^2 + \frac{1}{3}x \right) dx$$

$$= \frac{4}{5} \left( \frac{5}{18}x^3 + \frac{1}{6}x^2 \right) \bigg|_0^1 = \frac{4}{5} \left( \frac{5}{18} + \frac{1}{6} \right) = \frac{16}{45}$$

$$EX = EY = \frac{2}{5} \int_0^1 x(3x + 1) dx = \frac{2}{5} \left( x^3 + \frac{1}{2}x^2 \right) \bigg|_0^1 = \frac{2}{5} \cdot \frac{3}{2} = \frac{3}{5}$$

$$\text{Cov}(X, Y) = \frac{16}{45} - \left( \frac{3}{5} \right)^2 = \frac{80 - 81}{225} = -\frac{1}{225}$$

The correlation is the covariance of the standardized version of the random variables.

$$\rho_{X,Y} = E \left[ \left( \frac{X - \mu_X}{\sigma_X} \right) \left( \frac{Y - \mu_Y}{\sigma_Y} \right) \right] = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

In the example,

$$\sigma_X^2 = \frac{2}{5} \int_0^1 x^2(3x + 1) dx - \left( \frac{3}{5} \right)^2 = \frac{2}{5} \cdot \frac{13}{12} - \frac{9}{25} = \frac{11}{150}.$$
and

\[ \rho_{X,Y} = \frac{-1/225}{11/150} = -\frac{2}{33} = -0.06. \]

We can write equation (1) with \( a = 1 \) as

\[ \sigma^2_{X+cY} = \sigma^2_X + 2\rho_{X,Y}\sigma_X\sigma_Y c + \sigma^2_Yc^2. \]

This must be nonnegative for all values of \( c \). Thus, by considering the quadratic formula, we have that the discriminate

\[ 0 \geq (2\rho_{X,Y}\sigma_X\sigma_Y)^2 - 4\sigma^2_X\sigma^2_Y = (\rho^2_{X,Y} - 1)4\sigma^2_X\sigma^2_Y \quad \text{or} \quad \rho^2_{X,Y} \leq 1. \]

Consequently,

\[-1 \leq \rho_{X,Y} \leq 1.\]

When we have \( |\rho_{X,Y}| = 1 \), we also have for some value of \( c \) that

\[ \sigma^2_{X+cY} = 0. \]

In this case, \( X + cY \) is a constant random variable and \( X \) and \( Y \) are linearly related. In this case, the sign of \( \rho_{X,Y} \) depends on the sign of the linear relationship.

**Exercise 3.** \( \text{Var}(\sum_{i=1}^n a_i X_i) = \sum_{i=1}^n \sum_{j=1}^n a_i a_j \text{Cov}(X_i, X_j). \)

**Example 4** (variance of a hypergeometric). Consider an urn with \( B \) blue balls and \( G \) green balls. Remove \( K \) and let the random variable \( X \) denote the number of blue balls. Let

\[ X_i = \begin{cases} 0 & \text{if the } i\text{-th ball is green}, \\ 1 & \text{if the } i\text{-th ball is blue}. \end{cases} \]

Then, \( X = X_1 + X_2 + \cdots + X_K \). First, note that \( X_i \) is a Bernoulli random variable. \( EX_i = B/(B+G) \) and \( \text{Var}(X_i) = BG/(B+G)^2 \). Next, for the \( K(K-1) \) terms with \( i \neq j \),

\[ E[X_i X_j] = P\{X_i = 1, X_j = 1\} = P\{X_i = 1|X_j = 1\}P\{X_j = 1\} = \frac{B-1}{B+G-1} \cdot \frac{B}{B+G}. \]

Thus,

\[ \text{Cov}(X_i, X_j) = \frac{B(B-1)}{(B+G)(B+G-1)} - \left( \frac{B}{B+G} \right)^2 = \frac{B}{B+G} \left( \frac{B-1}{B+G-1} - \frac{B}{B+G} \right) = \frac{-BG}{(B+G)^2(B+G-1)}, \]

and using the formula in the previous exercise with the \( a_i = 1 \),

\[ \text{Var}(X) = K \frac{BG}{(B+G)^2} + K(K-1) \left( \frac{-BG}{(B+G)^2(B+G-1)} \right) = K \frac{BG}{(B+G)^2} \left( 1 - \frac{K-1}{B+G-1} \right). \]

To simplify the appearance of this expression, let \( N = K+G \) be the total number of balls and \( p = B/(B+G) \) be the proportion of the total number of balls that are blue. Then,

\[ \text{Var}(X) = Kp(1-p)\frac{N-K}{N-1}. \]

Note that if \( K << N \), then the variance is essentially the same as that of the corresponding binomial random variable. At the other extreme, if \( K = N \), then all the balls have been removed from the urn and \( \text{Var}(X) = 0. \)