# Power Series Solutions and Equations with Analytic Coefficients

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## 1 Form and Classification

We will move on the second order linear ordinary differential equations

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{1}$$

where $a$, $b$, and $c$ are analytical functions

The standard form (1) is

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

with

$$p(x) = \frac{b(x)}{a(x)} \quad \text{and} \quad q(x) = \frac{c(x)}{a(x)}.$$

A point $x_0$ is called an **ordinary point** of equation (2) if both $p$ and $q$ are analytic at $x_0$. If $x_0$ is not an ordinary point, it is called a **singular point**.

If $a$, $b$ and $c$ are analytic, then $x_0$ is a regular point as long as $a(x_0) \neq 0$. We can also include $x_0$ as an ordinary point if it has a **removable singularity**. For example, if $a(x) = x$ and $c(x) = \sin x$, then

$$q(x) = \frac{\sin x}{x}.$$

So the singular can be removed by defining at $x_0 = 0$

$$q(0) = \lim_{x \to 0} q(x) = 1.$$

and $q$ is analytic at 0.

By examining 2), we can see continue taking derivatives, giving equations in higher and higher order derivatives in $y$ as function of the lower order derivatives of $y$ and multiple derivatives of $p$ and $q$. Thus, we will use techniques based on the fact that the solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

can be realized as a power series solution about an ordinary point $x_0$.

## 2 First Order Example

For the first order differential equation,

$$y' + 2xy = 0,$$

the integrating factor is $\exp(x^2)$. So the solution

$$y(x) = A\exp(-x^2) = A\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n} = A\left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \frac{1}{24}x^8 - \cdots\right).$$

Now, let's look to develop the techniques that will lead to a series solution at $x_0 = 0$.

$$\begin{aligned} y(x) &= \textstyle\sum_{n=0}^{\infty} a_n x^n &&= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \cdots . \\ y'(x) &= \textstyle\sum_{n=0}^{\infty} n a_n x^{n-1} &&= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 \cdots . \\ 2xy(x) &= \textstyle\sum_{n=0}^{\infty} a_n x^{n+1} &&= 2a_0x + 2a_1x^2 + 2a_2x^3 + 2a_3x^4 \cdots . \end{aligned}$$

To make the summation more transparent, we shift the indexing on the sums so that the powers on $x$ match.

$$\begin{aligned} y'(x) &= \textstyle\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n &&= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 \cdots . \\ 2xy(x) &= \textstyle\sum_{n=1}^{\infty} a_{n-1}x^n &&= 0 + 2a_0x + 2a_1x^2 + 2a_2x^3 + 2a_3x^4 \cdots . \\ \hline 0 = y'(x) + 2xy(x) &= a_1 + \textstyle\sum_{n=1}^{\infty}((n+1)a_{n+1} + 2a_{n-1})x^n &&= a_1 + (2a_2 + 2a_0)x + (3a_3 + 2a_1)x^2 + (4a_4 + 2a_2)x^3 \end{aligned}$$

By the uniqueness of power series, each of the coefficients of $x_n$ on the right side must equal 0. Thus,

$$a_1 = 0, \quad 2a_2 + 2a_0 = 0, \quad 3a_3 + 2a_1 = 0 \quad 4a_4 + 2a_2 = 0 \quad \cdots \quad (n+1)a_{n+1} + 2a_{n-1} = 0$$

We have written the first few instances of the **recursion relation**

$$(n+1)a_{n+1} + 2a_{n-1} = 0$$

or again by shift the indices

$$(n+2)a_{n+2} + 2a_n = 0, \quad a_{n+2} = -\frac{2}{n+2}a_n.$$

Using the fact that $a_1 = 0$ and the sequence of equations for the coefficients, we see that $a_n = 0$ for all odd values of $n$. For the even values of $n$

$$\begin{aligned}
a_2 &= -\tfrac{2}{2}a_0 = -a_0 && = -a_0 \\
a_4 &= -\tfrac{2}{4}a_2 = -\tfrac{1}{2}a_2 && = \tfrac{1}{2}a_0 \\
a_6 &= -\tfrac{2}{6}a_4 = -\tfrac{1}{3}a_4 && = -\tfrac{1}{3\cdot 2}a_0 = \tfrac{1}{3!}a_0 \\
a_8 &= -\tfrac{2}{8}a_6 = -\tfrac{1}{4}a_6 && = \tfrac{1}{4\cdot 3\cdot 2}a_0 = -\tfrac{1}{4!}a_0 \\
\vdots &= \vdots = \vdots && = \vdots \\
a_{2k} &= -\tfrac{2}{2k}a_{2(k-1)} = -\tfrac{1}{k}a_{2(k-1)} && = (-1)^k\tfrac{1}{k\cdots 3\cdot 2}a_0 = (-1)^k\tfrac{1}{k!}a_0
\end{aligned}$$

# 3 Bessel Functions

Bessel functions were first defined by the mathematician Daniel Bernoulli and then generalized by Friedrich Bessel, as solutions

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0 \tag{3}$$

the parameter $\alpha$ is called the the order of the Bessel equation and function.

$$\begin{aligned}
y(x) &= \textstyle\sum_{n=0}^{\infty} a_n x^n && = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \cdots . \\
y'(x) &= \textstyle\sum_{n=0}^{\infty} n a_n x^{n-1} && = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 \cdots . \\
y''(x) &= \textstyle\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} && = 2\cdot 1a_2 + 3\cdot 2a_3x + 4\cdot 3a_4x^2 + 5\cdot 4a_5x^3 \cdots .
\end{aligned}$$

Let's look for a power series solution for the Bessel equation of order $\alpha = 0$.

$$\begin{aligned}
x^2y(x) &= \textstyle\sum_{n=0}^{\infty} a_n x^{n+2} && = a_0x^2 + a_1x^3 + a_2x^4 + \cdots . \\
xy'(x) &= \textstyle\sum_{n=0}^{\infty} n a_n x^{n} && = a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 \cdots . \\
x^2y''(x) &= \textstyle\sum_{n=0}^{\infty} n(n-1)a_n x^{n} && = 2\cdot 1a_2x^2 + 3\cdot 2a_3x^3 + 4\cdot 3a_4x^4 + \cdots .
\end{aligned}$$

Again, we shift the indexing on the sums so that the powers on $x$ match.

$$\begin{aligned}
x^2y(x) &= \textstyle\sum_{n=2}^{\infty} a_{n-2} x^{n} && = a_0x^2 + a_1x^3 + a_2x^4 + \cdots . \\
xy'(x) &= \textstyle\sum_{n=1}^{\infty} n a_n x^{n} && = a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 \cdots . \\
x^2y''(x) &= \textstyle\sum_{n=2}^{\infty} n(n-1)a_n x^{n} && = 2\cdot 1a_2x^2 + 3\cdot 2a_3x^3 + 4\cdot 3a_4x^4 + \cdots .
\end{aligned}$$

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$$
\begin{aligned}
0 &= x^2y''(x) + xy'(x) + x^2y(0) \\
&= a_1x + \textstyle\sum_{n=2}^{\infty}(a_{n-2} + na_n + n(n-1)a_{n-2})x^n \\
&= a_1x + \textstyle\sum_{n=2}^{\infty}(a_{n-2} + n^2a_n)x^n \\
&= a_1x + (a_0) + (a_0 + 2^2a_2)x^2 + (a_1 + 3^2a_3)x^3 + \cdots
\end{aligned}
$$

Thus,

$$
a_1 = 0, \quad a_0 + 2^2a_2 = 0, \quad a_1 + 3^2a_3 = 0, \quad a_2 + 4^2a_4 = 0, \quad \cdots \quad a_{n-2} + n^2a_n = 0.
$$

Again, the recursion relations ensures that $a_n = 0$ for odd values of $n$. For the even values of $n$

$$
\begin{aligned}
a_2 &= -\tfrac{1}{2^2}a_0 &&= -\tfrac{1}{2^2}a_0 \\
a_4 &= -\tfrac{1}{4^2}a_2 &&= \tfrac{1}{4^2\cdot 2^2}a_0 = \tfrac{1}{2^4(2\cdot 1)^2}a_0 \\
a_6 &= -\tfrac{1}{6^2}a_2 &&= \tfrac{1}{6^2\cdot 4^2\cdot 2^2}a_0 = \tfrac{1}{2^6(3\cdot 2\cdot 1)^2}a_0 \\
a_8 &= -\tfrac{2}{8}a_6 &&= \tfrac{1}{4\cdot 3\cdot 2}a_0 = -\tfrac{1}{4!}a_0 \\
\vdots &= \vdots &&= \vdots = \vdots \\
-a_{2k} &= -\tfrac{1}{k^2}a_{2(k-1)} &&= (-1)^k\tfrac{1}{2^{2k}(k!)^2}a_0
\end{aligned}
$$

In writing (3) in the standard form (2)

$$
y'' + \frac{1}{x}y' + \frac{x^2 - \alpha^2}{x^2}y = 0,
$$

we see that $x = 0$ is a singular point. This will lead to the fact that (3) will have solutions that are unbounded at $x = 0$. Bessel functions of the first kind, denoted as $J\alpha(x)$, are solutions of that are finite at the origin $x = 0$ for integer or positive $\alpha$. The usual form for $J_0(x)$, has $J_0(0) = 1$ and $J_0'(0) = 0$. Thus $a_0 = 1$ and $a_1 = 0$.

We can write the solution as

$$
J_0(x) = \sum_{k=0}^{\infty}(-1)^k\frac{1}{2^{2k}(k!)^2}x^{2k} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}.
$$

# 4 Equations with Analytic Coefficients

For $p$ and $q$ analytic function in equation (1) and $x_0$ an ordinary point for this equation. Then (1) has two linearly independent analytic solutions of the form

$$
y(x) = \sum_{n=0}^{\infty}a_n(x - x_0)^n.
$$

The radius of convergence of any power series solution is at least as large as the distance from $x_0$ to the nearest singular point (real or complex-valued).

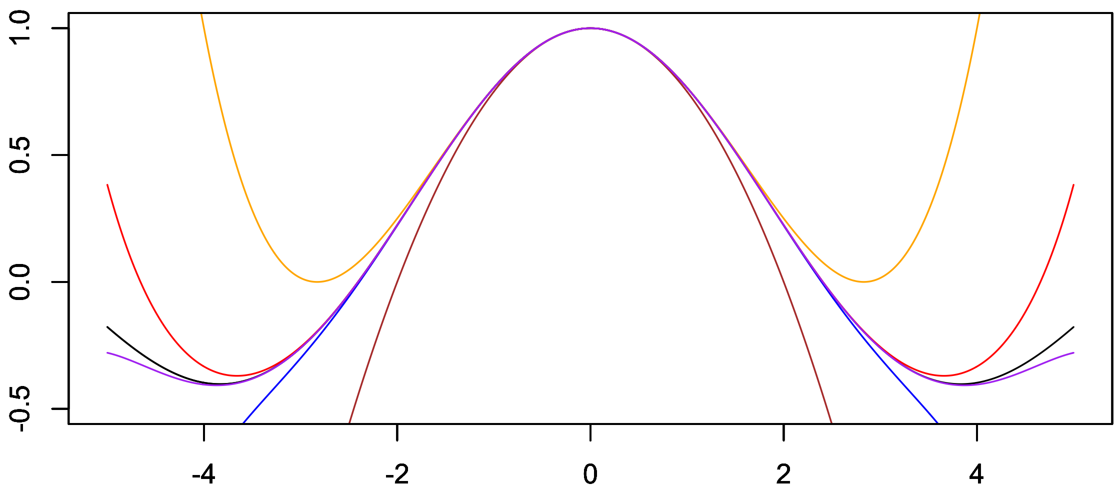


Figure 1: Taylor approximations of the Bessel function or order 0. Second degree in brown, 4th in orange, 6th in blue, 8th in red and 10th in purple.

**Exercise 1.** *Give the ensured radius of convergence about the ordinary point* $x_0 = 0$

- $y'' + 4x^2 y + y = 0$
- $(1 + x^2)y'' + 6xy + (\sin x)y = 0$
- $(1 + x)y'' + (1 - x^2)y + xy = 0$

We can carry out the procedure for any case in which $p$ and $q$ are analytic. The process can be involve if either $p$ or $q$ have a power series solutions with infinitely many terms. For example, take

$$y'' + e^x y' + (1 + x)y = 0,$$

then

$$\begin{aligned}
(1 + x)y(x) &= (1 + x)\sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty}(a_{n-1} + a_n)x^n. \\
e^x y'(x) &= \left(\sum_{n=1}^{\infty} n a_n x^n\right)\left(\sum_{n=0}^{\infty} \frac{1}{n!} x^n\right). \\
y''(x) &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}
\end{aligned}$$

The process is in principle the same - give the Cauchy product to $e^x y'(x)$, shift the summation indices, if necessary, set the coefficients of $x^n$ to zero and develop recursion relations.