Cauchy-Euler Equations and Method of Frobenius

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Certain singular equations have a solution that is a series expansion. We begin this investigation with **Cauchy-Euler equations**.

1 Cauchy-Euler Equations

A second order Cauchy-Euler equation has the form

$$ax^2y'' + bxy' + cy = 0 \tag{1}$$

for constants a, b, and c. Thus, $x_0 = 0$ is a singular point.

If we try a solution of the form

$$y(x) = x^r,$$

then

$$0 = ax^2y'' + bxy' + cy = (ar(r-1) + br + c)x^r = 0.$$

Thus x^r is a solution if

$$ar(r-1) + br + c = ar^2_(b-c)r + c = 0 \tag{2}$$

The equation (2) is called the **indicial** or **characteristic equation**.

Example 1. *For*

$$x^2y'' - 4xy' + 6y = 0,$$

the indicial equation is

$$0 = r(r-1) - 4r + 6 = r^2 - 5r + 6 = (r-2)(r-3).$$

and

$$y_1(x) = x^2 \quad \textit{and} \quad y_2(x) = x^3$$

are linearly independent solutions.

As with the constant coefficient equations, we have two additional considerations.

- If the roots in (2) are repeated, i.e., the indicial equation is $a(r-r_0)^2$, then the two solutions to (1) are

$$y_1(x) = x^{r_0} \quad \text{and} \quad y_2(x) = x^{r_0} \ln x.$$

- If the roots in (2) are complex conjugates, $\alpha \pm i\beta$, then the hen the two solutions to (1) are

$$y_1(x) = x^\alpha \cos(\beta \ln x) \quad \text{and} \quad y_2(x) = x^\alpha \sin(\beta \ln x).$$

Exercise 2. • *Show that*

$$y_(x) = c_1\sqrt{x} + c_2\sqrt{x}\ln x$$

$$4x'y'' + y = 0$$

- *Show that*

$$y(x) = \frac{1}{x}\left(c_1 \sin(2\ln x) + c_2 \cos(2\ln x)\right)$$

is a general solution to

$$x^2y'' + 3xy' + 5y = 0$$

This last equation show different behavior possible in Cauchy-Euler equations. The solutions are unbounded and oscillate more and more rapidly near $x = 0$.

2 Method of Frobenius

We moved from second order constant coefficient ordinary differential equations to differential equations having coefficients that are analytic functions of x. The **method of Frobenius** make a similar generalization from the Cauchy-Euler equations.

In this case, we start with

$$a(x)x^2y'' + b(x)xy' + c(x)y = 0 \tag{3}$$

and divide so that we have

$$\begin{aligned} y'' + \frac{b(x)}{xa(x)}y' + \frac{c(x)}{x^2a(x)}y &= 0 \\ y'' + p(x)y' + q(x)y &= 0 \end{aligned}$$

So,

$$xp(x) = \frac{b(x)}{a(x)} \quad \text{and} \quad x^2q(x) = \frac{c(x)}{a(x)}.$$

If we have the limits

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0}\frac{b(x)}{a(x)} = p_0 \quad \text{and} \quad \lim_{x\to 0} x^2q(x) = \lim_{x\to 0}\frac{c(x)}{a(x)} = q_0.$$

Then, for x near zero,

$$p(x) \approx xp_0 \quad \text{and} \quad q(x) \approx x^2q_0,$$

the solutions to (3) should be similar to the Cauchy-Euler equation

$$\begin{aligned} y'' + \frac{p_0}{x}y' + \frac{q_0}{x^2}y &= 0 \\ x^2y'' + xp_0y' + q_0y &= 0 \end{aligned} \tag{4}$$

We turn these observations into a definition.

A **singular point** x_0 of the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is said to be a **regular singular point** if both

$$(x - x_0)p(x) \quad \text{and} \quad (x - x_0)^2 q(x)$$

are analytic at x_0 Otherwise x_0 is called an **irregular singular point**.

Returning to (4), we again have an indicial equation

$$r(r-1) + p_0 r + q_0 = 0.$$

The roots of the indicial equation are called the **exponents** or **indices** of the singularity x_0.

Example 3. *For the Bessel differential equation*

$$\begin{aligned} x^2y'' + xy' + (x^2 - \alpha^2)y &= 0, \quad \alpha \geq 0 \\ y'' + \frac{1}{x}y' + \frac{x^2 - \alpha^2}{x^2}y &= 0 \end{aligned} \tag{5}$$

To see that $x_0 = 0$ is a regular singular point, note that

$$\lim_{x \to 0} xp(x) = 1 \quad \textit{and} \quad \lim_{x \to 0} x^2 q(x) = -\alpha^2.$$

The indicial equation is

$$0 = r(r-1) + r - \alpha^2 = r^2 - \alpha^2$$

So the roots are $\pm\alpha$

To begin, let's assume that this equation has distinct real roots, r_- and r_+. The method of Forbenius suggests that we look for solutions of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

For each of the roots to lead to distinct powers, the difference $r_+ - r_-$ cannot be an integer.

We apply this to Bessel's equation (5) by first writing series expansions for y, y' and y''.

$$\begin{aligned} y(x) &= \textstyle\sum_{n=0}^{\infty} a_n x^{n+\alpha} &&= a_0x^{\alpha} + a_1x^{1+\alpha} + a_2x^{2+\alpha} + a_3x^{3+\alpha} + a_4x^{4+\alpha} + \cdots. \\ y'(x) &= \textstyle\sum_{n=0}^{\infty} (n+\alpha)a_n x^{n+\alpha-1} &&= \alpha a_0 x^{\alpha-1} + (1+\alpha)a_1x^{\alpha} + (2+\alpha)a_2x^{1+\alpha} + (3+\alpha)a_3x^{2+\alpha} + \cdots. \\ y''(x) &= \textstyle\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1)a_n x^{n+\alpha-2} &&= \alpha(\alpha-1)a_0x^{\alpha-2} + (1+\alpha)\alpha a_1 x^{\alpha-1} + (2+\alpha)(1+\alpha)a_2x^{\alpha} + \cdots. \end{aligned}$$

For Bessel's equation

$$
\begin{aligned}
-\alpha^2 y(x) &= -\textstyle\sum_{n=0}^{\infty} \alpha^2 a_n x^{n+\alpha} && = -\alpha^2 a_0 x^{\alpha} - \alpha^2 a_1 x^{1+\alpha} - \alpha^2 a_2 x^{2+\alpha} - \alpha^2 a_3 x^{3+\alpha} - \alpha^2 a_4 x^{4+\alpha} + \cdots. \\
x^2 y(x) &= \textstyle\sum_{n=0}^{\infty} a_n x^{n+\alpha+2} && = a_0 x^{2+\alpha} + a_1 x^{3+\alpha} + a_2 x^{4+\alpha} + a_3 x^{5+\alpha} + \cdots. \\
x y'(x) &= \textstyle\sum_{n=0}^{\infty} (n+\alpha) a_n x^{n+\alpha} && = \alpha a_0 x^{\alpha} + (1+\alpha) a_1 x^{1+\alpha} + (2+\alpha) a_2 x^{2+\alpha} + (3+\alpha) a_3 x^{4+\alpha} + \cdots. \\
x^2 y''(x) &= \textstyle\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha} && = \alpha(\alpha-1) a_0 x^{\alpha} + (1+\alpha)\alpha a_1 x^{\alpha+1} + (2+\alpha)(1+\alpha) a_2 x^{\alpha+2} + \cdots.
\end{aligned}
$$

To line up the indices, we adjust the expression for the $x^2 y(x)$ term.

$$
x^2 y(x) \quad = \textstyle\sum_{n=2}^{\infty} a_{n-2} x^{n+\alpha} \quad = a_0 x^{2+\alpha} + a_1 x^{3+\alpha} + a_2 x^{4+\alpha} + a_3 x^{5+\alpha} + \cdots.
$$

and add. Notice that the sum on the x^{α} term is

$$
-\alpha^2 a_0 + \alpha a_0 + \alpha(\alpha-1) a_0 = 0
$$

. So, a_0 is arbitrary. For the $x^{\alpha+1}$ term

$$
-\alpha^2 a_1 + (1+\alpha) a_1 + (1+\alpha)\alpha a_1 = 0, \quad (2\alpha+1) x_1 = 0, \quad \text{and} \quad a_1 = 0.
$$

For the powers of $x^{n+\alpha}$ we have that

$$
\begin{aligned}
0 &= (n+\alpha)(n+\alpha-1) a_n + (n+\alpha) a_n - \alpha^2 a_n + a_{n+2} \\
&= ((n-\alpha)^2 - \alpha^2 a)_n + a_{n+2} \\
&= (n+2\alpha) n a_n + a_{n-2}
\end{aligned}
$$

From this we see that the terms a_n for n odd vanish. Also, notice this recursion relation agrees with the case $\alpha = 0$ determined earlier.

For the even values of n

$$
\begin{aligned}
a_2 &= -\tfrac{1}{2(2+2\alpha)} a_0 && = -\tfrac{1}{2(2+2\alpha)} a_0 \\
a_4 &= -\tfrac{1}{4(4+2\alpha)} a_2 && = \tfrac{1}{4(4+2\alpha)\cdot 2(2+2\alpha)} a_0 = \tfrac{1}{2^4 (2\cdot 1)(2+\alpha)(1+\alpha} a_0 \\
a_6 &= -\tfrac{1}{6(6+2\alpha)} a_2 && = \tfrac{1}{6(6+2\alpha)\cdot 4(4+2\alpha)\cdot 2(2+2\alpha)} a_0 = \tfrac{1}{2^6 (3\cdot)2\cdot 1)(3+\alpha)(2+\alpha)(1+\alpha} a_0 \\
a_8 &= -\tfrac{2}{8} a_6 && = \tfrac{1}{4\cdot 3\cdot 2} a_0 = -\tfrac{1}{4!} a_0 \\
\vdots &= \vdots && = \vdots = \vdots \\
-a_{2k} &= -\tfrac{1}{k^2} a_{2(k-1)} && = (-1)^k \tfrac{1}{2^{2k} k! (k+\alpha)_k} a_0
\end{aligned}
$$

The term $(a)_k = a(a-1)\cdots(a-k+1)$ is called the falling factorial and is read "x falling k".

Thus, $a_0 J_{\alpha}(x)$ is a solution to (5) where

$$
J_{\alpha}(x) = \sum_{n=0}^{\infty} (-1)^k \frac{1}{2^{2k} k! (k+\alpha)_k} x^{2k} = \sum_{n=0}^{\infty} (-1)^k \frac{1}{k! (k+\alpha)_k} \left(\frac{x}{2}\right)^{2k}
$$