Calculus on Matrices and First Order Systems

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1 Differentation and Integration

The usual calculus operations when applied to matrices mean applying the operation to each entry of the matrix.

For a vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix},$$

the derivative

$$\mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_d' \end{pmatrix},$$

For a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rc} \end{pmatrix},$$

the integral

$$\int A(t)dt = \begin{pmatrix} \int a_{11}(t)dt & \int a_{12}(t)dt & \cdots & \int a_{1c}(t)dt \\ \int a_{21}(t)dt & \int a_{22}(t)dt & \cdots & \int a_{2c}(t)dt \\ \vdots & \vdots & \ddots & \vdots \\ \int a_{r1}(t)dt & \int a_{r2}(t)dt & \cdots & \int a_{rc}(t)dt \end{pmatrix},$$

Exercise 1. *For*

$$A(t) = \begin{pmatrix} e^t & \cos 2t \\ 1/t & t^2+1 \end{pmatrix},$$

find $A'(t)$ *and* $\int A(t)dt$

Exercise 2. *Let*

$$A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

then

$$\mathbf{x} = \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}$$

is a solution to $\mathbf{x}' = A\mathbf{x}$.

Exercise 3. *Let*

$$A = \begin{pmatrix} k_1 & 0 \\ -0 & k_2 \end{pmatrix}$$

then

$$\mathbf{x} = \begin{pmatrix} \exp(k_1 t) \\ \exp(k_2 t) \end{pmatrix}$$

is a solution to $\mathbf{x}' = A\mathbf{x}$.

Differentiation Formulas for Matrix Functions For matrics A, B and C,

- If C a constant matrix, then
$$\frac{d(CA)}{dt} = C\frac{dA}{dt}$$
- (linearity)
$$\frac{d(A+B)}{dt} = \frac{dA}{dt} + \frac{dB}{dt}$$
- (product rule)
$$\frac{d(AB)}{dt} = A\frac{dB}{dt} + \frac{dA}{dt}B$$

2 Normal Form

A system of d linear differential equations is said to be in normal form if it is expressed as

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t) \tag{1}$$

where, for each t,

- $\mathbf{x}(t)$ and $\mathbf{f}(t)$ are d-dimensional column vectors
- $A(t)$ is a $d \times d$ matrix.

As with a scalar linear differential equation, a system is called

- **homogeneous** when $\mathbf{f}(t) = 0$; otherwise, it is called **nonhomogeneous**.
- The homogeneous equation is linear and so any linear combination of solutions is also a solution.
- **constant coefficient** if all the entries of A are all constants.

Many properties we have for systems follow from ideas in our previous discussions on differential equations.

- **Initial Value Problem**
  The **initial value problem** for the normal system (1) is the problem of finding a differentiable vector-valued function $\mathbf{x}(t)$ that satisfies the system (1) on an interval I with given the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, for some $t_0 \in I$.

- **Existence and Uniqueness**
  Suppose $A(t)$ and $\mathbf{x}(t)$ are continuous on an open interval I that contains the point t_0. Then, for any choice of the initial vector $\mathbf{x}_0$, there exists a unique solution $\mathbf{x}(t)$ on the whole interval I to the initial value problem.

- **Linear Independence**
  The vector-valued functions $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_k(t)$ are **linearly independent** if no linear combination of these equations is equal to the zero function. Otherwise, $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_k(t)$ are **linearly dependent**. In this case we can find constants $c_1, c_2, \ldots c_k$ not all of which are zero so that

  $$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_k\mathbf{x}_k(t) = 0.$$

  Exercise 4. *Show that the two vectors*

  $$\mathbf{x}_1(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \textit{and} \quad \mathbf{x}_2(t) = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2}$$

  are linearly independent.

  We can check for linear independence by turning the d d-dimensional column vectors into a $d \times d$ matrix

  $$X(t) = (\mathbf{x}_1(t)\ \mathbf{x}_2(t)\ \cdots, \mathbf{x}_d(t)).$$

  The determinant of this matrix is called the **Wornskian**, and is written $W(\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_d(t))$

- **Solutions in the Homogeneous Case**
  Let $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_d(t)$ be linearly independent solutions to the homogeneous system,

  $$\mathbf{x}'(t) = A(t)\mathbf{x}(t) \tag{3}$$

  on the interval I, then every solution to (3) can be expressed in the form

  $$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_k\mathbf{x}_k(t).$$

  Exercise 5. *Let $A(t)$ be the constant matrix*

  $$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

  then $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions to (3).

- **Fundamental Solution**
  In the case that the Wronskian does not valnish on the interval I, then the matrix $X(t)$ is called the fundamental matrix for (3). Thus for a column vector of constants, $\mathbf{c} = (c_1, c_2, \ldots, c_d)$, we can write,

  $$\mathbf{x}(t) = X(t)\mathbf{c}.$$

Exercise 6. *Fr the constant matrix $A(t)$ above be the constant matrix*

$$X(t) = \begin{pmatrix} e^{3t} & e{-}t \\ e^{3t} & -e^{-t} \end{pmatrix}$$

is the fundamental solution.

The Wronskian is $W(\mathbf{x}_1(t), \mathbf{x}_2(t)) = -e^{3t}e^{-t} - e^{-t}e^{3t} = -e^{2t} \neq 0$ for all t. Thus, the solutions are linearly independent.

For initial condition given by the column vector $\mathbf{x}_0 = (x_0, x_1)$

$$\mathbf{x}_0 = X(0)\mathbf{c} \quad \text{and} \quad \mathbf{c} = X(0)^{-1}\mathbf{x}_0.$$

Exercise 7. *Find the inverse of*

$$X(0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

- **Solutions in the Inhomogeneous Case**
  Let $\mathbf{x}_p$ be a **particular solution** to the nonhomogeneous system (1) and let $(\mathbf{x}_1(t)\ \mathbf{x}_2(t)\ \cdots, \mathbf{x}_d(t))$ a fundamental solutions to the homogeneous system (1), the every solution to (3) on I can be expressed

$$\mathbf{x}(t) = \mathbf{x}_p(t) + c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \ldots + c_k\mathbf{x}_d(t).$$