The order statistics
\[ X(1), X(2), \ldots, X(n) \]
is the increasing ordered arrangement of the sample
\[ X_1, X_2, \ldots, X_n. \]

The sample range \( R = X(n) - X(1) \). The sample median
\[ M = \begin{cases} X((n+1)/2) & \text{if } n \text{ is odd,} \\ X(n/2) + X((n/2)+1) & \text{if } n \text{ is even.} \end{cases} \]

Fix a value \( x \) and consider the Bernoulli trials
\[ Y_1 = I_{(-\infty,x]}(X_1), Y_2 = I_{(-\infty,x]}(X_2), \ldots, Y_n = I_{(-\infty,x]}(X_n). \]

If \( F \) is the common cumulative distribution function for the \( X_i \), then
\[ p = P\{Y = 1\} = P\{X_1 \leq x\} = F(x). \]

Let \( S_n = Y_1 + Y_2 + \cdots + Y_n \), then \( S_n \geq j \) if and only if at least \( j \) of the \( X_i \leq x \). In turn, this is true if and only if the \( j \)-th order statistic \( X(j) \leq x \). Consequently,
\[ P\{X(j) \leq x\} = P\{S_n \geq j\} = \sum_{k=j}^{n} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=j}^{n} \binom{n}{k} F(x)^k (1-F(x))^{n-k}. \]

For \( X_i \) discrete random variables, the mass function,
\[ f_{X(j)}(x) = P\{X(j) = x\} = P\{X(j) \leq x\} - P\{X(j) < x\} = \sum_{k=j}^{n} \binom{n}{k} \left( F(x)^k (1-F(x))^{n-k} - F(x-)^k (1-F(x-))^{n-k} \right). \]

For \( X_i \) continuous random variables having common density function \( f \), the density function,
\[ f_{X(j)}(x) = \frac{d}{dx} \left( \sum_{k=j}^{n} \binom{n}{k} F(x)^k (1-F(x))^{n-k} \right). \]
Exercise 1.

\[
\binom{n}{k+1}(k+1) = \binom{n}{k}(n-k).
\]

Differentiating the individual terms

\[
\frac{d}{dx} \left( \binom{n}{k} F(x)^k (1 - F(x))^{n-k} \right) = \binom{n}{k} k F(x)^{k-1} (1 - F(x))^{n-k} - (n-k) F(x)^k (1 - F(x))^{n-k-1} f(x)
\]

\[
= \binom{n}{k} k F(x)^{k-1} (1 - F(x))^{n-k} f(x) - \binom{n}{k+1} (k+1) F(x)^k (1 - F(x))^{n-k-1} f(x)
\]

Summing the first term, we obtain

\[
\sum_{k=j}^{n} \binom{n}{k} k F(x)^{k-1} (1 - F(x))^{n-k} f(x).
\]

Summing the second term, we obtain

\[
\sum_{k=j}^{n} \binom{n}{k+1} (k+1) F(x)^k (1 - F(x))^{n-k-1} f(x) = \sum_{k=j+1}^{n+1} \binom{n}{k} k F(x)^{k-1} (1 - F(x))^{n-k} f(x).
\]

The \(k = n + 1\) term in the second sum is zero. Thus, the difference leaves only the first term in the first sum.

\[
f_{X_{(j)}}(x) = \binom{n}{j} j F(x)^{j-1} (1 - F(x))^{n-j} f(x).
\]

If the \(X_i\) are uniform random variables on the interval \([0, 1]\), then

\[
f_{X_{(j)}}(x) = \binom{n}{j} j x^{j-1} (1 - x)^{n-j}, \quad 0 \leq x \leq 1,
\]

a beta random variable with parameters \(\alpha = j\) and \(\beta = n - j + 1\).

For continuous random variables, the probability that two random variables take on the same value

\[
P\{X_i = X_j, \text{ for some } i \neq j\} = 0.
\]

To find the joint density of the order statistics, write \(\pi\) for an arbitrary permutation on \(\{1, 2, \ldots, n\}\). Each of these \(n!\) permutations of an equal chance of being the order statistics for an given sample. Indeed, the mapping from the observations to the order statistics is an \(n!\) to one map. In addition, the permutation matrix associated with \(\pi\) has determinant \(\pm 1\) depending on whether or not the permutation is even or odd. Because this is a linear map, the Jacobian \(J_{\pi}\) of the inverse transformation is equal to the inverse of the permutation matrix.

For example, assume that \(X_1 = X_{(2)}, X_2 = X_{(3)}, X_3 = X_{(1)}\). The inverse transformation

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
X_{(1)} \\
X_{(2)} \\
X_{(3)}
\end{bmatrix}.
\]

The matrix is the Jacobian of the inverse transformation.
Write \( B_\pi = \{ X_{\pi(1)} < X_{\pi(2)} < \cdots < X_{\pi(n)} \} \). Then,

\[
\begin{align*}
  f_{x_{(1)}, x_{(2)}, \ldots, x_{(n)}}(x_1, x_2, \ldots, x_n) &= \sum_{\pi} |\det(J_\pi)| f(x_{\pi(1)}) f(x_{\pi(2)}) \cdots f(x_{\pi(n)}) = n! f(x_1) f(x_2) \cdots f(x_n),
\end{align*}
\]

\( x_{(1)} < x_{(2)} < \cdots < x_{(n)} \)

**Example 2.** Integrating to find the lower dimensional marginals is an arduous task. We shall show that we can use a hierarchical model strategy to compute the density. We show this in the case of two order statistics and the uniform distribution on \([0, 1]\).

Write, for \( i < j \),

\[
  f_{x_{(i)}, x_{(j)}}(u,v) = f_{x_{(i)|x_{(j)}}}(u|v) f_{x_{(j)}}(v)
\]

From the computation above, we know that

\[
  f_{x_{(j)}}(v) = \binom{n}{j} v^{j-1} (1 - v)^{n-j}, \quad 0 \leq v \leq 1,
\]

Given that \( X_{(j)} = v \), we have \( j - 1 \) random variables uniformly distributed on \([0, v]\). Among these random variables, we are looking for the \( i \)th order statistics. Now,

\[
  F_X(u|X \leq v) = u, \quad \text{for } 0 \leq u \leq v.
\]

Thus,

\[
  f_{x_{(i)|x_{(j)}}}(u|v) = \binom{j - 1}{i} \left( \frac{u}{v} \right)^{i-1} \left( \frac{v - u}{v} \right)^{j-i-1}.
\]

Now multiply

\[
  f_{x_{(i)}, x_{(j)}}(u,v) = \binom{j - 1}{i} \left( \frac{u}{v} \right)^{i-1} \left( \frac{v - u}{v} \right)^{j-i-1} \cdot \binom{n}{j} v^{j-1} (1 - v)^{n-j}
\]

\[
  = \binom{n}{j} \binom{j - 1}{i} i u^{i-1} (v - u)^{j-i-1} (1 - v)^{n-j}
\]

\[
  = \frac{n!}{(j - 1)! (n - j)! (i - 1)! (j - i - 1)!} u^{i-1} (v - u)^{j-i-1} (1 - v)^{n-j}
\]

Note that the coefficient is a trinomial coefficient. This (correctly) suggests that an approach using a trinomial distribution similar to the strategy used above with the binomial distribution will succeed.