Convergence Concepts

November 17, 2009

Definition 1. *Let $X, X_1, X_2, \cdots$ be a sequence of random variables.*

1. *We say that X_n* **converges to** *X* **almost surely** *($X_n \to^{a.s.} X$) if*

$$P\{\lim_{n\to\infty} X_n = X\} = 1.$$

2. *We say that X_n* **converges to** *X* **in** *L^p or in p***-th moment***, $p > 0$, ($X_n \to^{L^p} X$) if,*

$$\lim_{n\to\infty} E[|X_n - X|^p] = 0.$$

3. *We say that X_n* **converges to** *X* **in probability** *($X_n \to^{P} X$) if, for every $\epsilon > 0$,*

$$\lim_{n\to\infty} P\{|X_n - X| > \epsilon\} = 0.$$

4. *We say that X_n* **converges to** *X* **in distribution** *($X_n \to^{\mathcal{D}} X$) if, for every bounded continuous function $h : \mathbb{R} \to \mathbb{R}$.*

$$\lim_{n\to\infty} Eh(X_n) = Eh(X).$$

For almost sure convergence, convergence in probability and convergence in distribution, if X_n converges to X and if g is a continuous then $g(X_n)$ converges to $g(X)$.

Convergence in distribution differs from the other modes of convergence in that it is based not on a direct comparison of the random variables X_n with X but rather on a comparision of the distributions $P\{X_n \in A\}$ and $P\{X \in A\}$. Using the change of variables formula, convergence in distribution can be written

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h(x)\, dF_{X_n}(x) = \int_{-\infty}^{\infty} h(x)\ dF_X(x).$$

We can use this to show that $X_n \to^{\mathcal{D}} X$ if and only if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

for all points x that are continuity points of F_x.

Example 2. *Let X_n be uniform on the points $\{1/n, 2/n, \cdots n/n = 1\}$. Then, using the convergence of a Riemann sum to a Riemann integral, we have as $n \to \infty$,*

$$Eh(X_n) = \sum_{i=1}^{n} h\left(\frac{i}{n}\right)\frac{i}{n} \to \int_0^1 h(x)\,dx = Eh(X)$$

where X is a uniform random variable on the interval $[0, 1]$.

Example 3. *Let $X_i, i \geq 1$, be independent uniform random variable in the interval $[0, 1]$. and let $Y_n = n(1 - X_{(n)})$. Then,*

$$F_{Y_n}(y) = P\{n(1 - X_{(n)} \leq y\} = P\left\{1 - \frac{y}{n} \leq X_{(n)}\right\} = 1 - \left(1 - \frac{y}{n}\right)^n \to 1 - e^{-y}.$$

Thus, the magnified gap between the highest order statistic and 1 converges in distribution to an exponentail random variable.

1 Relationships among the Modes of Convergence

We begin with Jensen's inequality.

One way to characterize a convex function ϕ is that its graph lies above any tangent line. If we look at the point $x = \mu$, then this statement becomes

$$\phi(x) - \phi(\mu) \geq \phi'(\mu)(x - \mu).$$

Now replace x with the random variable X having mean μ and take expectations.

$$E[\phi(X) - \phi(\mu)] \geq E[\phi'(\mu)(X - \mu)] = \phi'(\mu)E[X - \mu] = 0.$$

Consequently,

$$E\phi(X) \geq \phi(EX) \tag{1}$$

The expression in (1) is known as **Jensen's inequality**.

- By examining Chebychev's inequality,

  $$P\{|X_n - X| > \epsilon\} \leq \frac{E[|X_n - X|^p]}{\epsilon^p}$$

  we see that convergence in L^p implies convergence in probability.

- If $q > p$, then $\phi(x) = x^{q/p}$ is convex and by Jensen's inequality

  $$E|X|^q = E|X|^{p(q/p)} \geq (E|X|^p)^{q/p}.$$

  We can also write this

  $$(E|X|^q)^{1/q} \geq (E|X|^p)^{1/p}.$$

  From this, we see that q-th moment convergence implies p-th moment convergence.

- Also, convergence almost surely implies convergence in probability.

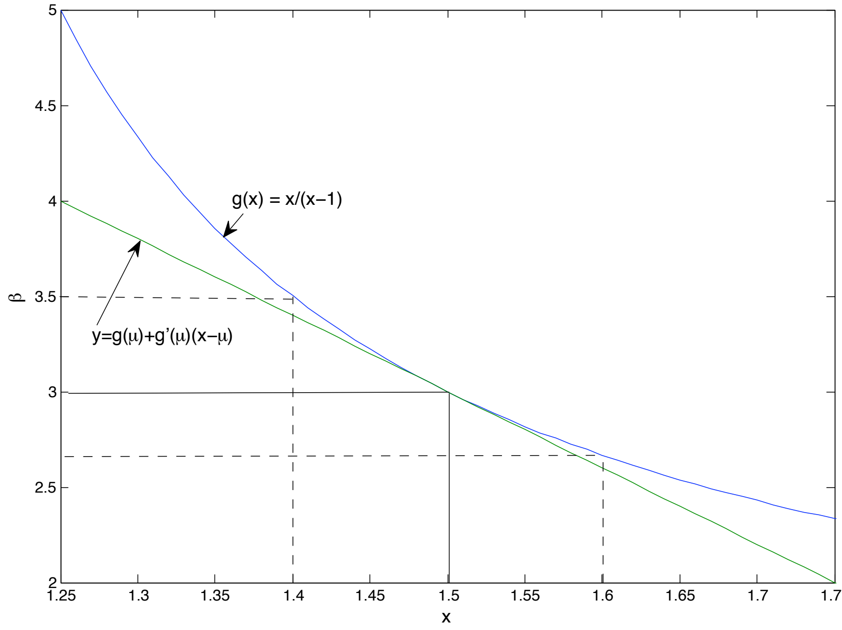
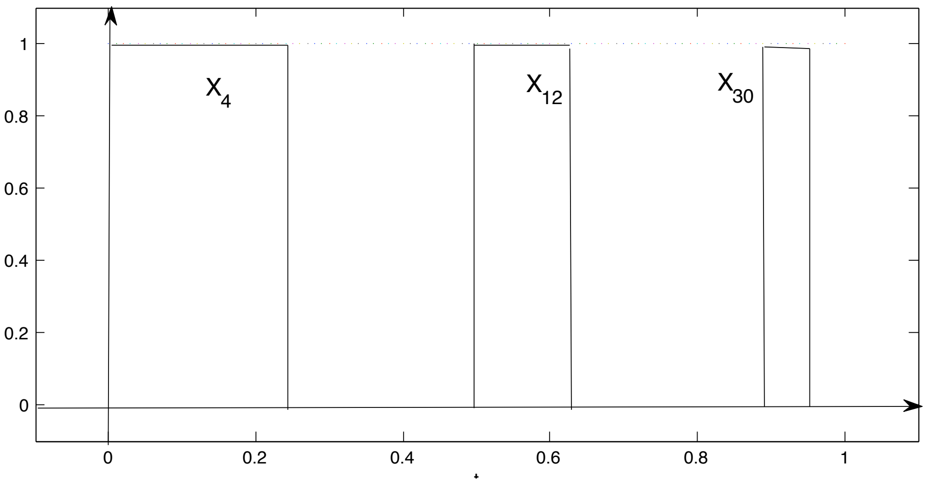


Figure 1: Graph of a convex function. Note that the tangent line is below the graph of g.

- Convergence in probability implies convergence in distribution.
- We can write any positive integer by $2^n + k$, $k = 0, 1, \ldots, 2^{n-1} - 1$ and define

$$X_{2^n+k}(s) = I_{(k2^{-n},(k+1)2^{-n}]}(s), \quad 0 \le s \le 1.$$



Then, if our probability space $[0,1]$ has a probability that assigns its length to an interval, then for $0 < \epsilon < 1$,

$$P\{|X_{2^n+k}| > \epsilon\} = 2^{-n}$$

and the sequence converges to 0 in probability. However, for each $s \in (0,1]$, $X_j(s) = 1$ and $X_j(s) = 0$ for infinitely many j and so the sequence does not converge almost surely.

- $E|X_{2^n+k} - 0|^p = 2^{-np}$, so the sequence converges in L^p to 0.
- If

  $$Y_{2^n+k}(s) = 2^n I_{((k-1)2^{-n},k2^{-n}]}(s), \quad 0 \le s \le 1.$$

  Then, again,

  $$P\{|Y_{2^n+k}| > \epsilon\} = 2^{-n}$$

  and the sequence converges to 0 in probability. However,
- $E|Y_{2^n+k} - 0| = 2^n P\{Y_{2^n+k} = 2^n\} = 2^n 2^{-n} = 1$, so the sequence does not converge in L^1.

2 Laws of Large Numbers

The best convergence theorem showing that the sample mean converges to the mean of the common distribution is the **strong law of large numbers**

Theorem 4. *Let $X_1, X_2, \ldots$ be independent identically distributed random variables and set $S_n = X_1 + \cdots + X_n$, then*

$$\lim_{n\to\infty} \frac{1}{n} S_n$$

exists almost surely if and only if $E|X_1| < \infty$. In this case the limit is $EX_1 = \mu$ with probability 1.

Convergence laws using a mode other than almost sure is called **a weak law**. Here is a L^2-weak law of large numbers.

Theorem 5. *Assume that $X_1, X_2, \ldots$ for a sequence of real-valued uncorrelated random variable with common mean μ. Futher assume that their variances are bounded by some constant C. Write*

$$S_n = X_1 + \cdots + X_n.$$

Then

$$\frac{1}{n} S_n \to^{L^2} \mu.$$

Proof. Note that $E[S_n/n] = \mu$. Then

$$E[(\frac{1}{n}S_n - \mu)^2] = \mathrm{Var}(\frac{1}{n}S_n) = \frac{1}{n^2}(\mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)) \le \frac{1}{n^2}Cn.$$

Now, let $n \to \infty$ $\square$

Because L^2 convergence implies convergence in probability, we have, in addition,

$$\frac{1}{n}S_n \to^P \mu.$$

Exercise 6. *For the triangular array* $\{X_{n,k}; 1 \le n, 1 \le k \le k_n\}$. *Let* $S_n = X_{n,1} + \cdots + X_{n,k_n}$ *be the* n*-th row rum. Assume that* $ES_n = \mu_n$ *and that* $\sigma_n^2 = Var(S_n)$. *If*

$$\frac{\sigma_n^2}{b_n^2} \to 0 \textit{ then } \frac{S_n - \mu_n}{b_n} \to^{L^2} 0.$$

Example 7.

*(*Coupon Collectors Problem) *Let* $Y_1, Y_2, \ldots,$ *be independent random variables uniformly distributed on* $\{1, 2, \ldots, n\}$ *(sampling with replacement). Define the random sequence* $T_{n,k}$ *to be minimum time* m *such that the cardinality of the range of* $(Y_1, \ldots, Y_m)$ *is* k. *Thus,* $T_{n,0} = 0$. *Define the triangular array*

$$X_{n,k} = T_{n,k} - T_{n,k-1},\ k = 1, \ldots, n.$$

For each n, $X_{k,n} - 1$ *are independent* $Geo(1 - (k-1)/n)$ *random variables. Therefore*

$$EX_{n,k} = (1 - \frac{k-1}{n})^{-1} = \frac{n}{n-k-1}, \quad Var(X_{n,k}) = \frac{(k-1)/n}{((n-k-1)/n)^2}.$$

Consequently, for $T_{n,n}$ *the first time that all numbers are sampled,*

$$ET_{n,n} = \sum_{k=1}^{n} \frac{n}{n-k-1} = \sum_{k=1}^{n} \frac{n}{k} \approx n \log n, \quad Var(T_{n,n}) = \sum_{k=1}^{n} \frac{(k-1)/n}{((n-k-1)/n)^2} \le \sum_{k=1}^{n} \frac{n^2}{k^2}.$$

By taking $b_n = n \log n$, *we have that*

$$\frac{T_{n,n} - \sum_{k=1}^{n} \frac{n}{k}}{n \log n} \to^{L^2} 0$$

and

$$\frac{T_{n,n}}{n \log n} \to^{L^2} 1.$$