Homogeneous Liner Systems with Constant Coefficients

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The object of study in this section is

$$\mathbf{x}'(t) = A\mathbf{x}(t) \tag{1}$$

where A is a $d \times d$ constant matrix whose entries are real numbers.

As before, we will look to the exponential function for solutions.

$$\mathbf{x}(t) = e^{rt}\mathbf{u}$$

where $\mathbf{u} \neq 0$ is a d-dimensional column vector and r is a number. Substituting into (1), we find that

$$re^{rt}\mathbf{u} = Ae^{rt}\mathbf{u} = e^{rt}A\mathbf{u}$$

Therefore,

$$r\mathbf{u} = A\mathbf{u} \quad \text{and} \quad (A - rI)\mathbf{u} = 0$$

For $\mathbf{u} \neq 0$, we must have non-trivial solutions to this algebraic equation and consequently the determinant

$$\det(A - rI) = 0.$$

Let's give names to the concepts that this calculations generate. For a $d \times d$ constant matrix A,

- The **eigenvalues** or **characteristic values** of A are those (possibly complex) values r for which

  $$(A - rI)\mathbf{u} = 0$$

  has at least one non-trivial solution.

- The corresponding solutions $\mathbf{u}$ are called the **eigenvectors** or **characteristic vectors** of A associated with r. Note that any non-zero scalar multiple of a eigenvector is an eigenvector.

- The determinant $p(r) = \det(A - rI)$ is a polynomial of degree d. $p(r)$ is called the **characteristic polynomial** of A and $p(r) = 0$ is called the **characteristic equation** of A.

We will see that ehe characteristic equation plays a role for systems similar to the role played by the auxiliary equation for scalar equations.

Example 1. *Let $A(t)$ be the constant matrix*

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

the characteristic polynomial

$$p(r) = \det(A - rI) = \det \begin{pmatrix} 1-r & 2 \\ 2 & 1-r \end{pmatrix} = (1-r)^2 - 4$$

The characteristic equation

$$(1-r)^2 - 4 = 0$$

has solutions

$$r_1 = -1 \quad \text{and} \quad r_2 = 3$$

To find eigenvectors, we find non-trivial solutions to

$$\begin{pmatrix} 1-r_1 & 2 \\ 2 & 1-r_1 \end{pmatrix} \mathbf{u}_1 = 0 \quad \text{and} \quad \begin{pmatrix} 1-r_2 & 2 \\ 2 & 1-r_2 \end{pmatrix} \mathbf{u}_2 = 0,$$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \mathbf{u}_1 = 0 \quad \text{and} \quad \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{u}_2 = 0,$$

Thus, solutions are

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus,

$$\mathbf{x}(t) = c_1 e^t \mathbf{u}_1 + c_2 e^{3t} \mathbf{u}_2$$

is a general solution.

Exercise 2. *Find a general solution to (1) for*

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}$$

Give the fundamental matrix and determine the solution that satisfies

$$\mathbf{x}(0) = \mathbf{x}_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

We now give an examplw with $d = 3$.

Example 3. *To determine a general solution to (1) for*

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}$$

we first compute the characteristic polynomial.

$$\begin{aligned}
p(r) &= \det(A - rI) = \det \begin{pmatrix} -r & 1 & 1 \\ 1 & -2-r & -3 \\ -1 & 1 & 2-r \end{pmatrix} \\
&= -r \cdot \det \begin{pmatrix} -2-r & -3 \\ 1 & 2-r \end{pmatrix} - 1 \cdot \det \begin{pmatrix} 1 & -3 \\ -1 & 2-r \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 1 & -2-r \\ -1 & 1 \end{pmatrix} \\
&= -r\left((-2-r)(2-r) - (-3)\right) - 1((2-r) - (-3)(-1)) + (1 - (-2-r)(-1)) \\
&= -r(-4 + r^2 + 3) - (2 - r - 3) + (1 - 2 - r) = -r(r^2 - 1) + (r+1) - (r+1) = -r(r^2 - 1) \\
&= -r(r-1)(r+1)
\end{aligned}$$

The characteristic equation $p(r) = 0$ has roots,

$$r_1 = -1, \quad r_2 = 0, \quad \textit{and} \quad r_3 = 1$$

For the eigenvectors,

$$(A - r_1 I) = \begin{pmatrix} -r_1 & 1 & 1 \\ 1 & -2 - r_1 & -3 \\ -1 & 1 & 2 - r_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ -1 & 1 & 3 \end{pmatrix}$$

Check that we can take $\mathbf{u}_1 = (1 \ -2 \ 1)$.

For $r_2 = 0$, $(A - r_2 I) = A$. Check that we can take $\mathbf{u}_2 = (1 \ -1 \ 1)$ For the eigenvectors,

$$(A - r_3 I) = \begin{pmatrix} -r_3 & 1 & 1 \\ 1 & -2 - r_3 & -3 \\ -1 & 1 & 2 - r_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -3 & -3 \\ -1 & 1 & 1 \end{pmatrix}$$

Check that we can take $\mathbf{u}_3 = (0 \ 1 \ -1)$. The general soluiion is

$$\mathbf{x}(t) = e^{-t}\mathbf{u}_1 + \mathbf{u}_2 + c_3 e^{-t}\mathbf{u}_3.$$

1 Complex Eigenvalues

To start with some notation. With $z = a + ib$ is a complex number, the we write the **complex conjugate** $\bar{z} = a - ib$.

Exercise 4.

- *A constant c is real if and only if $\bar{c} = c$*
- $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$.

A constant a is real is to have $\bar{a} = a$. We extend this notation to vectors and matrices. Thus, a matrix A has real-valued entries if and only if $\bar{A} = A$.

Suppose that two solutions to the characteristic equation $p(r) = 0$ are $r_\pm = \alpha \pm i\beta$. Thus, $r_- = \bar{r}_+$.

If $\mathbf{u} = \mathbf{a} + i\mathbf{b}$ is an eigenvector for A with eigenvalue r_+. Here $\mathbf{a}$ and $\mathbf{b}$ are real-valued vectors. Then,

$$0 = (A - r_+ I)(\mathbf{a} + i\mathbf{b}).$$

Now take the complex conjugate of this equation, noting that $\bar{A} = A$ and $\bar{I} = I$.

$$0 = \overline{(A - r_+ I)(\mathbf{a} + i\mathbf{b})} = (A - r_- I)(\mathbf{a} - i\mathbf{b}),$$

showing that $\mathbf{a} - i\mathbf{b}$ is an eigenvector for A with eigenvalue r_-. Thus, we have two linearly independent solutions

$$\begin{aligned}
\tilde{\mathbf{x}}_1(t) &= e^{(\alpha+i\beta)t}(\mathbf{a}+i\mathbf{b}) \\
&= e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{a}+i\mathbf{b}) \\
&= e^{\alpha t}((\cos\beta t\ \mathbf{a} - \sin\beta t\ \mathbf{b}) + i(\sin\beta t\ \mathbf{a} + \cos\beta t\ \mathbf{b})) \\
&= \mathbf{x}_1(t) + i\mathbf{x}_2(t) \\
\tilde{\mathbf{x}}_2(t) &= e^{(\alpha-i\beta)t}(\mathbf{a}-i\mathbf{b}) \\
&= e^{\alpha t}(\cos\beta t - i\sin\beta t)(\mathbf{a}-i\mathbf{b}) \\
&= e^{\alpha t}((\cos\beta t\ \mathbf{a} - \sin\beta t\ \mathbf{b}) - i(\sin\beta t\ \mathbf{a} + \cos\beta t\ \mathbf{b})) \\
&= \mathbf{x}_1(t) - i\mathbf{x}_2(t)
\end{aligned}$$

where

$$\mathbf{x}_1(t) = e^{\alpha t}((\cos\beta t\ \mathbf{a} + \sin\beta t\ \mathbf{b}) \quad \text{and} \quad \mathbf{x}_2(t) = e^{\alpha t}((\cos\beta t\ \mathbf{a} - \sin\beta t\ \mathbf{b}).$$

Now $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linear combinations of $\tilde{\mathbf{x}}_1(t)$ and $\tilde{\mathbf{x}}_2(t)$, and thus are solutions to (1)
Note that $\mathbf{x}_1(t)$ is the real part of $\tilde{\mathbf{x}}_1(t)$ and that $\mathbf{x}_2(t)$ is the imaginary part.

Exercise 5. *Check that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent*

Example 6. *Find a general solution to (1) for*

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$$

the characteristic polynomial

$$p(r) = \det(A-rI) = \det\begin{pmatrix} 3-r & -2 \\ 4 & -1-r \end{pmatrix} = (3-r)(-1-r)+8 = r^2-2r-3+8 = r^2-2r+5 = (r-1)^2+4.$$

Thus, the root of the characteristic equation

$$r_\pm = 1 \pm 2i.$$

As noted above, we need only find the eigenvector to r_+,

$$0 = \begin{pmatrix} 3-r_+ & -2 \\ 4 & -1-r_+ \end{pmatrix}\mathbf{z} = \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\mathbf{z}$$

We have an eigenvector

$$\mathbf{z} = \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{a} + i\mathbf{b}.$$

$$\mathbf{x}_1(t) = e^t((\cos 2t\ \mathbf{a} + \sin 2t\ \mathbf{b}) \quad \textit{and} \quad \mathbf{x}_2(t) = e^t((\cos 2t\ \mathbf{a} - \sin 2t\ \mathbf{b}).$$

$$\mathbf{x}_1(t) = \begin{pmatrix} e^t(\cos 2t + \sin 2t) \\ 2e^t\cos 2t \end{pmatrix} \quad \textit{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} e^t(\cos 2t - \sin 2t) \\ 2e^t\cos 2t \end{pmatrix}.$$

The fundamental solution is

$$X(t) = \begin{pmatrix} e^t(\cos 2t + \sin 2t) & e^t(\cos 2t - \sin 2t) \\ 2e^t \cos 2t & 2e^t \cos 2t \end{pmatrix}$$

Notice that the Wronskian

$$W(\mathbf{x}_1, \mathbf{x}_2) = e^t(\cos 2t + \sin 2t) \cdot 2e^t \cos 2t - e^t(\cos 2t - \sin 2t) \cdot 2e^t \cos 2t = 4e^{2t} \sin 2t \cos 2t.$$