Using Green’s functions with inhomogeneous BCs

Surprise: Although Green’s functions satisfy homogeneous boundary conditions, they can be used for problems with inhomogeneous BCs!

For self-adjoint $L$ and $u, v$ with homogeneous boundary conditions it follows that

$$\int_{\Omega} (L v) u \, dx - \int_{\Omega} (L u) v \, dx = 0.$$ 

But if $u, v$ don’t satisfy homogeneous boundary conditions, get

$$\int_{\Omega} (L v) u \, dx - \int_{\Omega} (L u) v \, dx = \text{boundary terms involving } u \text{ and } v.$$ 

This is called the Green’s formula, which depends on $L$ and $\Omega$. 
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Collapsing the integral involving the \( \delta \) function,

\[
u(x_0) = \int_{\Omega} G(x, x_0) f(x) \, dx + \text{boundary terms}\]
Green’s formula for Laplacian

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$$\Delta u = f \text{ in } \Omega, \quad u = h \text{ on } \partial\Omega,$$

For dimensions $\geq 2$, the Green’s formula is just Green’s identity

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Substituting $v(x) = G(x, x_0)$ into Green’s formula,

$$\int_{\Omega} u(x)\delta(x-x_0) - G(x, x_0) f(x) \, dx = \int_{\partial \Omega} u(x) \nabla_x G(x, x_0) \cdot \hat{n}(x) - G(x, x_0) \nabla u(x) \cdot \hat{n}(x) \, dx.$$
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\[ \int_{\Omega} u(x) \delta(x-x_0) - G(x, x_0) f(x) \, dx = \int_{\partial \Omega} u(x) \nabla_x G(x, x_0) \cdot \hat{n}(x) - G(x, x_0) \nabla u(x) \cdot \hat{n}(x) \, dx \]

Simplifies to

\[ u(x_0) = \int_{\Omega} G(x, x_0) f(x) \, dx + \int_{\partial \Omega} h(x) \nabla_x G(x, x_0) \cdot \hat{n}(x) \, dx, \]
In the case that $\Omega$ is a disk of radius $a$, Green’s function is

$$G(r, \theta; r_0, \theta_0) = \frac{1}{4\pi} \ln \left( \frac{a^2}{r_0^2} \frac{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)}{r^2 + a^4/r_0^2 - 2ra^2/r_0 \cos(\theta - \theta_0)} \right).$$
Example: the Poisson integral formula revisited

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The boundary value problem

$$\Delta u = 0, \quad u(a, \theta) = h(\theta)$$

has a solution

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Need normal derivative of $G$

$$\nabla_x G(x, x_0) \cdot \hat{n}(x) = G_r(r, \theta; r_0, \theta_0)$$

$$= \frac{1}{4\pi} \left( \frac{2r_0 - 2r \cos(\theta - \theta_0)}{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)} - \frac{2r_0r^2 - 2ra^2 \cos(\theta - \theta_0)}{r^2r_0^2 + a^4 - 2rr_0a^2 \cos(\theta - \theta_0)} \right),$$

which at $r = a$ is

$$\frac{a}{2\pi} \left( \frac{1 - (r/a)^2}{r^2 + a^2 - 2ar \cos(\theta - \theta_0)} \right).$$
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Parameterize boundary integral using θ and \(|dx| = a \, d\theta\),

\[
u(r_0, \theta_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2 - r_0^2)h(\theta)}{a^2 + r_0^2 - 2ar_0 \cos(\theta - \theta_0)} \, d\theta.
\]
Neumann boundary conditions

Want to solve

\[ \nabla^2 u = 0, \quad \lim_{z \to \infty} u(x, y, z) = 0, \quad u_z(x, y, 0) = h(x, y), \]

in upper half space \( \{(x, y, z)|z > 0\} \).
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Green’s formula

\[ \int_{\Omega} u \Delta v - v \Delta u \, dx = \int_{\partial \Omega} u \nabla v \cdot \hat{n} - v \nabla u \cdot \hat{n} \, dx. \]

has both Dirichlet and Neumann boundary terms in \( u \), but only know \( \nabla u(x, y, 0) \cdot \hat{n} = -u_z(x, y, 0) \).
Want to solve

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To make \( \nabla v \cdot \hat{n} = \nabla G \cdot \hat{n} \) vanish on boundary, need Green’s function to respect “boundary condition principle”:

The Green’s function must have the same type of boundary conditions as the problem to be solved, and they must be homogeneous.
Neumann boundary conditions, cont.

Method of images prescribes *even* reflection so $G_z = 0$ when $z = 0$:

$$G(x, y, z; x_0, y_0, z_0) = G_3(x, y, z; x_0, y_0, z_0) + G_3(x, y, z; x_0, y_0, -z_0)$$

$$= \frac{1}{4\pi} \left( \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} + \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z+z_0)^2}} \right)$$
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Substituting \( v(x) = G(x, x_0) \) into Green's formula and collapsing the \( \delta \)-function integral,

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u(x_0) = -\int_{\partial \Omega} G(x; x_0) \nabla u(x) \cdot \hat{n}(x) \, dx,
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Neumann boundary conditions, cont.

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Since \( \hat{n} \) is directed *outward*, \( \nabla u(x) \cdot \hat{n}(x) = -u_z(x, y, 0) \), and

\[
u(x_0, y_0, z_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{h(x, y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + z_0^2}} \, dx \, dy.
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$$G(x, x_0) = G(x_0, x), \quad \text{"Reciprocity"}$$
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Proof: Insert $v(x) = G(x, x_1)$, $u(x) = G(x, x_2)$, $\mathcal{L}v = \delta(x - x_1)$ and $\mathcal{L}u = \delta(x - x_2)$ into Green’s formula:

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which simplifies to $G(x_1, x_2) - G(x_2, x_1) = 0$. 
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Also: can interchange arguments of partial derivatives, e.g.

$$\partial_x G(x, x_0) = \lim_{h \to 0} \frac{G(x + h, x_0) - G(x, x_0)}{h}$$

$$= \lim_{h \to 0} \frac{G(x_0, x + h) - G(x_0, x)}{h}$$

$$= \partial_{x_0} G(x_0, x).$$
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For example, representation formula

$$u(x_0) = \int_{\Omega} G(x, x_0) f(x) dx + \int_{\partial\Omega} h(x) \nabla_x G(x, x_0) \cdot \hat{n}(x) \, dx,$$

can be rewritten by exchanging the notation for $x$ and $x_0$ and using reciprocity,

$$u(x) = \int_{\Omega} G(x, x_0) f(x_0) dx_0 + \int_{\partial\Omega} h(x_0) \nabla_{x_0} G(x, x_0) \cdot \hat{n}(x_0) \, dx_0.$$