Method of Characteristics

Recall the first order, linear wave equation and initial condition for $u(x,t)$

$$u_t + cu_x = 0, \quad u(x,0) = f(x), \quad -\infty < x < \infty.$$

One way to interpret this is that the directional derivative in the direction $(c,1)$ is zero. Thus u is constant along curves (which happen to be straight lines here) of the form $x - ct =$ constant. Therefore to determine the value of u at (x,t), we go backward along this line until we get to $t=0$, and then determine the value of u from the initial condition.

We can carry out this same idea for equations of the form

$$u_t + c(x,t)u_x = 0, \quad u(x,0) = f(x), \quad -\infty < x < \infty. \tag{1}$$

Let $x = X(t)$ be any "trajectory". How does u evolve along this trajectory?

$$\frac{d}{dt}u(X(t),t) = X'(t)u_x + u_t,$$

by the chain rule. If we happen to pick $X'(t) = c(X(t),t)$, then

$$\frac{d}{dt}u(X(t),t) = c(X(t),t)u_x(X(t),t) + u_t(X(t),t) = 0$$

by using equation (1). Thus u is constant along ALL curves which are solutions of the ODE $X'(t) = c(X,t)$. To solve for u at some (x,t), we go "backward" along this curve until we hit time zero, and since u is constant along this curve, we find that the value of u is determined by the initial condition. In other words, if $X(t) = x$, then $u(x,t) = u(X(0),0) = f(X(0))$.

The curves $X(\tau)$ that solve the "final value" problem

$$X'(\tau) = c(x,\tau), \quad X(t) = x, \tag{2}$$

are called *characteristics*. Notice that we use the variable τ to parameterize the curve, since "t" denotes the time at which the characteristic curve "ends".

We can even be more general, solving an inhomogeneous equation

$$u_t + c(x,t)u_x = g(u,x,t), \quad u(x,0) = f(x), \quad -\infty < x < \infty. \tag{3}$$

The only difference between this and (1) is that u is not constant along characteristics, but rather solves an ODE

$$u_\tau = g(u, X(\tau), \tau), \tag{4}$$

subject to the initial condition $u(\tau = 0) = u(X(0)) = f(X(0))$. Thus, to find u at some point (x, t), we go *backwards* along the characteristic that ends at x until time zero, then solve the ODE (4) *forwards* until $\tau = t$.

Let us summarize this into an algorithm:

1. Solve $X'(\tau) = c(X, t)$ with "final" condition $X(t) = x$. The solution will depend on x, thus we can write $X = X(\tau; x)$.

2. Solve $u'(\tau) = g(u, X(\tau; x), \tau)$ subject to initial condition $u(0) = u(X(0; x))$.

3. Finally, the solution is

$$u(x, t) = u(\tau = t).$$

   Notice that since the ODE for u depends on x via its initial condition, so will the final answer.

Examples...