The Method of Characteristics

Recall that the first order linear wave equation
\[ u_t + cu_x = 0, \quad u(x, 0) = f(x) \]
is constant in the direction \((1, c)\) in the \((t, x)\)-plane, and is therefore constant on lines of the form \(x - ct = x_0\). To determine the value of \(u\) at \((x, t)\), we go backward along these lines until we get to \(t = 0\), and then determine the value of \(u\) from the initial condition. The result is \(u(x, t) = u(x - ct, 0) = f(x - ct)\).

There are many extensions to this simple idea. We begin by describing the situation for linear and nearly linear equations.

1 Homogeneous transport equations

We can carry out this same idea for equations of the form
\[ u_t + c(x, t)u_x = 0, \quad u(x, 0) = f(x), \quad -\infty < x < \infty. \tag{1} \]
Let \(X(T)\) be any “trajectory” - think of it as a curve in the \((x, t)\) plane, where \((X, T)\) are supposed to be the \((x, t)\) coordinates. How does \(u\) evolve as we move along this trajectory?

\[ \frac{d}{dT} u(X(T), T) = X'(T)u_x(X(T), T) + u_t(X(T), T) \]
by the chain rule. If we happen to pick \(X'(T) = c(X(T), T)\), then

\[ \frac{d}{dT} u(X(T), T) = c(X(T), T)u_x(X(T), T) + u_t(X(T), T) = 0 \]
by virtue of equation (1). Thus \(u\) is constant along ALL curves which are solutions of the ODE \(X'(T) = c(X, T)\). To solve for \(u\) at some \((x, t)\), we go “backward” along this curve until we hit time zero, and since \(u\) is constant along this curve, we find that the value of \(u\) is determined by the initial condition. In other words, if \(X(t) = x\), then \(u(x, t) = u(X(0), 0) = f(X(0))\).

The curves \(X(T)\) that solve the ODE
\[ X'(T) = c(X, T), \quad X(t) = x, \tag{2} \]
are called characteristics. For the purpose of finding characteristics, \((x, t)\) are fixed constants, and it is \(X\) and \(T\) that vary along characteristics.
**Example 1.** Solve $u_t + xu_x = 0$ with initial condition $u(x, 0) = \cos(x)$.

**Solution.** A characteristic curve ending at $(x, t)$ will solve

$$X'(T) = X(T), \quad X(t) = x,$$

whose solution is $X(T) = x \exp(T - t)$. Since $u$ is constant along the characteristic,

$$u(x, t) = u(X(0), 0) = \cos(xe^{-t}).$$

**Example 2.** We want to solve

$$yu_x = xu_y, \quad u(0, y) = 2y^2 \text{ for } y > 0$$

**Solution.** This equation can be written in the form (1) as

$$u_x - \frac{x}{y}u_y = 0,$$

treating $x$ like the time variable. Let $Y(X)$ denote characteristic curves, which is a solution to

$$Y'(X) - \frac{X}{Y} = 0.$$

Separating variables $Y dY = -X dX$ leads to $X^2 + Y^2 = C$; in other words, characteristics are closed curves encircling the origin. If an implicitly defined characteristic curve passes through $(x, y)$, it is described by $X^2 + Y^2 = x^2 + y^2$. Since the solution is constant along this curve, setting $X = 0$ and using the side condition in (3) gives

$$u(x, y) = u(X, Y) = 2Y^2 = 2(x^2 + y^2).$$

Notice that if a boundary condition were imposed on the entire $y$-axis, then characteristic curves would intersect this boundary both at $(0, y)$ and $(0, -y)$. Unless $u(0, y) = u(0, -y)$, this problem would not have a solution.

### 1.1 Inhomogeneous transport equations

We can also solve equations of the form

$$u_t + c(x, t)u_x = g(u, x, t), \quad u(x, 0) = f(x), \quad -\infty < x < \infty.$$  

The only difference between this and equation (1) is that $u$ is not constant along characteristics, but evolves according to

$$\frac{d}{dt} u(X(t), t) = g(u, X(t), t).$$
In other words, if we let $U(T) = u(X(T), T)$ be the solution restricted to a single characteristic, it solves an initial value problem, namely

$$U'(T) = g(U, X(T), T), \quad U(0) = u(X(0), 0) = f(X(0)).$$

Thus, to find $u$ at some point $(x, t)$, we go backwards along the characteristic that ends at $x$ until time zero, then solve the ODE (5) forwards until $T = t$.

### 1.2 The method of characteristics for linear problems

We can summarize ideas above as an algorithm:

1. **Find the characteristic terminating at** $(x, t)$: Solve $X'(T) = c(X, T)$ with the “final” condition $X(t) = x$. Note that the solution for $X(T)$ will depend on $x$ and $t$ as parameters.

2. **Determine the solution along a characteristic**: Solve $U''(T) = g(U, X(T), T)$ subject to initial condition $U(0) = U(X(0), T)$. Again the solution depends on $x$ and $t$ as parameters.

3. **Find the solution at the endpoint of the characteristic**: The solution of the PDE at $(x, t)$ is simply $u(x, t) = U(t)$.

Here are a couple examples of how this is used.

**Example 1.** Solve

$$u_t + (x + t)u_x = t, \quad u(x, 0) = f(x).$$

**Solution.** Characteristic curves solve the ODE

$$X'(T) = X + T, \quad X(t) = x.$$  

This equation has a particular solution, $X_p = -T - 1$; the general solution is therefore $X(T) = Ce^T - T - 1$. Using the condition $X(t) = x$, we find that

$$X(T) = e^{T-t}(x + t + 1) - T - 1.$$  

Now we need to find how $u$ changes along the characteristic. We solve

$$U'(T) = T, \quad U(0) = f(X(0)) = f(e^{-t}(x + t + 1) - 1).$$

whose solution by direct integration is

$$U(T) = f(e^{-t}(x + t + 1) - 1) + \frac{1}{2} T^2.$$
Finally, the solution at \((x, t)\) is simply the value at the endpoint of the characteristic
\[
 u(x, t) = U(t) = f(e^{-t}(x + t + 1) - 1) + \frac{1}{2}t^2.
\]

**Example 2.** Solve the nonlinear problem
\[
 u_t + 3u_x = -u^2, \quad u(x, 0) = f(x).
\]

**Solution.** In this case, characteristics solve \(X'(t) = 2\) with \(X(t) = x\). Along each characteristic, the solution evolves as \(U'(t) = -U^2(t)\) with \(U(0) = f(X(0)) = f(-2t + x)\). This is nonlinear, but we can solve it since it is just a separable ODE which can be written \(dU/U^2 = -dT\), so that integration gives \(1/U = T + C\). Using the initial condition, one gets \(C = \frac{1}{f(x-2t)}\) and
\[
 U(T) = \frac{1}{T + \frac{1}{f(x-2t)}}.
\]

The final solution is obtained by setting \(u(x, t) = U(t)\).

**Example 3.** Suppose water flows over a landscape whose elevation is described by \(h(x)\). A simple model for surface water flow says that the flow velocity is equal (in the right units) to \(-h'(x)\). It follows that if \(u(x, t)\) is the depth of water, then the flux of \(u\) is \(J = h'(x)u\). In the absence of sources \(u\) satisfies the conservation equation \(u_t + (-h'(x)u)_x = 0\), which can be written in the form (4) as
\[
 u_t - h'(x)u_x = h''(x)u. \tag{6}
\]

The term on the right accounts for the fact that water will accumulate in valleys where \(h'' > 0\), and is depleted from hills where \(h'' < 0\).

Consider a simple model for a valley where \(h = x^2\), and suppose that the initial depth is localized as
\[
 u(x, 0) = \begin{cases} 
 1 & |x| \leq 1 \\
 0 & |x| > 1 
\end{cases}
\]

Since equation (6) reads \(u_t - 2xu_x = 2u\), characteristics solve \(X'(t) = -2X\) together with the terminal condition \(X(t) = x\). The solution of this problem is
\[
 X(T) = xe^{2(t-T)}.
\]
The solution on characteristics now solves $U' = 2U$ with initial condition

$$U(0) = \begin{cases} 1 & |X(0)| \leq 1 \\ 0 & |X(0)| > 1 \end{cases}$$

Therefore $U(T) = e^{2T}$ if $|X(0)| = |xe^{2t}| < 1$, or zero otherwise. It follows that when $t = T$,

$$u(x,t) = \begin{cases} e^{2t} & |x| \leq e^{-2t} \\ 0 & |x| > e^{-2t} \end{cases}$$

The depth of the fluid layer therefore increases exponentially, but its width decreases exponentially in a way such that $\int u \, dx$ remains constant.