Green's functions, formulas and representations

Suppose that we want to solve a linear, inhomogeneous equation of the form

$$\mathcal{L}u(\mathbf{x}) = f(\mathbf{x}) \quad + \text{homogeneous boundary conditions.} \tag{1}$$

Here u, f are functions whose domain is Ω, which could be either a finite or infinite domain in any dimension. The boundary conditions could be Dirichlet, Neumann, or some mixture. In this discussion , $\mathcal{L}$ will be self-adjoint with respect to the usual L^2 inner product and the imposed homogeneous boundary conditions. We will assume that equation (1) admits a solution for EVERY f. This is not the case for all linear operators – for those that do not admit unique solutions, there is an extended notion of *generalized Green's functions* which we do not pursue here.

1 The delta function and distributions

There is a great need in differential equations to define objects that arise as limits of functions and behave like functions under integration but are not, properly speaking, functions themselves. These objects are sometimes called *generalized* functions, or *distributions*. The most basic one of these is the δ-"function".

For each $\epsilon > 0$, define the family of ordinary functions

$$\delta_\epsilon(x) = \frac{1}{\epsilon\sqrt{\pi}} e^{-x^2/\epsilon^2}. \tag{2}$$

When ϵ is small, the graph of δ_ϵ is essentially just a spike of unit integral at $x = 0$. Thus for any continuous $f(x)$,

$$\int_{-\infty}^{\infty} f(x)\delta_\epsilon(x - x_0)dx \approx f(x_0)$$

for small ϵ. On the other hand, taking the limit $\epsilon \to 0$ *inside* the integral makes no sense: the limit of δ_ϵ is not a function at all! To get around this, we define the object δ to act as follows:

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0).$$

This not only defines what δ is, but defines what it means to integrate when the integrand has a δ-function.

In general, all distributions can be approximated by smooth, ordinary functions. Thus one can think of operations involving distributions as just limits of the same operations involving those ordinary functions. In fact, many of the usual things one does with functions, including differentiation and integral transforms, are well defined for distributions precisely in this way.

There are δ functions for higher dimensions also. We define the n-dimensional δ function as

$$\int_{\mathbb{R}^n} f(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_0)dx^n = f(\mathbf{x}_0),$$

where $\mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^n$. Sometimes we write this higher dimensional delta function as a product of one dimensional ones $\delta(\mathbf{x}) = \delta(x)\cdot\delta(y)\cdot\ldots$.

1.1 Distributions as derivatives

One useful aspect of distributions is that they make sense of derivatives of functions which are non-smooth, or even unbounded. Suppose that $g(x)$ cannot be differentiated everywhere in its domain. It does make sense, however, to talk about the *integrals* involving g'. Let $\phi(x)$ be any differentiable function with $\phi(\pm\infty) = 0$. By formal integration by parts we define

$$\int_{-\infty}^{\infty} g'(x)\phi(x)dx \equiv -\int_{-\infty}^{\infty} g(x)\phi'(x)dx.$$

Notice that the expression on the right makes perfect sense. We define the *distributional derivative* of $g(x)$ to be the distribution $d(x)$ so that

$$\int_{-\infty}^{\infty} d(x)\phi(x)dx = -\int_{-\infty}^{\infty} g(x)\phi'(x)dx.$$

for every smooth ϕ. For example, if $H(x)$ is the step function which is zero when $x < 0$ and one when $x > 0$, one can verify that

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)dx = -\int_{-\infty}^{\infty} H(x)\phi'(x)dx.$$

since both the left and right hand sides are equal to $\phi(0)$. Thus the delta function is the distributional derivative of the unit step function.

In higher dimensions, one can make similar definitions of distributional derivatives by using Green's formulas. For example, the statement $\Delta u(\mathbf{x}) =$

$\delta(\mathbf{x})$ means that for smooth $\phi(\mathbf{x})$ and any open domain D,

$$\begin{aligned}\phi(0) = \int_D \phi(\mathbf{x})\delta(\mathbf{x})dx^n = \int_D \phi(\mathbf{x})\Delta u(\mathbf{x})dx^n = \\ \int_D u(\mathbf{x})\Delta\phi(\mathbf{x})dx^n + \int_{\partial D} \phi(\mathbf{x})\nabla u(\mathbf{x})\cdot\hat{\mathbf{n}} - u(\mathbf{x})\nabla\phi(\mathbf{x})\cdot\hat{\mathbf{n}}\,dx^{n-1}.\end{aligned} \tag{3}$$

Even if $u(\mathbf{x})$ is unbounded at $\mathbf{x} = 0$ (which is usually the case), the integrals on the right can make perfect sense. In particular, if $\phi = 1$ then a useful fact is that

$$1 = \int_{\partial D} \nabla u(\mathbf{x})\cdot\hat{\mathbf{n}}\,dx^{n-1}. \tag{4}$$

Similar formulas are available for other operators by integrating $\mathcal{L}u = \delta(x)$ and moving derivatives off of u by application of the divergence theorem and/or Green's identities.

2 Green's functions

If equation (1) has a solution of the form

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x};\mathbf{x}_0)f(\mathbf{x}_0)d\mathbf{x}_0 \tag{5}$$

for every source function $f(x)$, then the function $G(\mathbf{x};\mathbf{x}_0)$ is called the Green's function. Is there only one such function G so that (5) holds? Suppose that there is another function G_2 so that

$$u(\mathbf{x}) = \int_\Omega G_2(\mathbf{x};\mathbf{x}_0)f(\mathbf{x}_0)d\mathbf{x}_0 \tag{6}$$

Subtracting (5) and (6) gives

$$0 = \int_\Omega \Big(G(\mathbf{x};\mathbf{x}_0) - G_2(\mathbf{x};\mathbf{x}_0)\Big)f(\mathbf{x}_0)d\mathbf{x}_0, \tag{7}$$

for every function f, which means that $G - G_2$ (as a function of x_0) is orthogonal to every function, so that $G - G_2 = 0$. Thus the Green's function is unique.

2.1 Relationship to the delta function

Part of the problem with the definition (5) is that it doesn't tell how to construct G. It is useful to imagine what happens when f is a point source at $\mathbf{x}_i \in \Omega$, i.e. $f(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_i)$. Plugging into (5) we learn that the solution to

$$\mathcal{L}u(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_i) \quad + \text{homogeneous boundary conditions.} \tag{8}$$

is just

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}; \mathbf{x}_0)\delta(\mathbf{x}_0 - \mathbf{x}_i)d\mathbf{x}_0 = G(\mathbf{x}; \mathbf{x}_i). \tag{9}$$

In other words, we find that the Green's function $G(\mathbf{x}; \mathbf{x}_0)$ formally satisfies

$$\mathcal{L}_x G(\mathbf{x}; \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) \tag{10}$$

(the subscript means that the linear operator acts on $\mathbf{x}$, not $\mathbf{x}_0$). Equation (9) gives us intuition of how to think about Green's functions. In words, it says that $G(\mathbf{x}; \mathbf{x}_0)$ is the *influence* felt at point $\mathbf{x}$ due to a *source* at point $\mathbf{x}_0$.

Equation (10) is a more useful way of defining G since we can in many cases solve this "almost" homogeneous equation, either by direct integration or using Fourier techniques. In particular, (10) can be rewritten as two conditions

$$\mathcal{L}_x G(\mathbf{x}, \mathbf{x}_0) = 0, \quad \text{when } \mathbf{x} \neq \mathbf{x}_0 \tag{11}$$

and integrating (formally) both sizes of $\mathcal{L}_x G(\mathbf{x}; x_0) = \delta(\mathbf{x} - \mathbf{x}_0)$

$$\int_B \mathcal{L}_x G(\mathbf{x}, \mathbf{x}_0)dx = 1, \quad \text{for any ball } B \text{ centered at } \mathbf{x}_0. \tag{12}$$

where the integral is meant in the distributional sense (i.e. by integration by parts). Equation (11) is a homogeneous equation with a "hole" in the domain at $\mathbf{x}_0$. Equation (12) is a normalization (or in 1-D, called a "jump" condition) to get the "size" of the singularity at $\mathbf{x}_0$ correct. In addition to (11-12), G must also satisfy the same homogeneous boundary conditions.

2.2 Three basic examples

We now specialize to the case of $\mathcal{L} = \Delta$.
One dimension. Suppose $u : \mathbb{R} \to \mathbb{R}$ solves the (ordinary differential equation)

$$u_{xx} = f, u(0) = 0 = u(L). \tag{13}$$

The corresponding Green's function will solve

$$G_{xx}(x;x_0) = 0 \text{ for } x \neq x_0, \quad G(0,x_0) = 0 = G(L,x_0), \tag{14}$$

The normalization condition reads

$$\int_{x_0-r}^{x_0+r} G''(x;x_0)dx = 1$$

for any r such that the interval of integration is in $[0,L]$. One can show (as an exercise) that G is continuous but that it has a jump in its derivative at $x = x_0$:

$$\lim_{x\to x_0^+} G_x(x;x_0) - \lim_{x\to x_0^-} G_x(x;x_0) = 1.$$

The piecewise solution to (14) is

$$G(x;x_0) = \begin{cases} c_1 x & x < x_0 \\ c_2(x-L) & x > x_0 \end{cases} \tag{15}$$

Imposing continuity at $x = x_0$ and the jump condition gives

$$c_1 x_0 = c_2(x_0 - L), \quad c_2 - c_1 = 1,$$

so that $c_1 = (x_0 - L)/L$ and $c_2 = x_0/L$. It follows that the solution to (13) can be written using G as

$$u(x) = \frac{1}{L}\left(\int_0^x x_0(x-L)f(x_0)dx_0 + \int_x^L x(x_0-L)f(x_0)dx_0\right).$$

Three dimensions. Now suppose $u : \mathbb{R}^3 \to \mathbb{R}$ solves

$$\Delta u = f, \quad \lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0. \tag{16}$$

In this case the homogeneous "boundary" condition is actually a far-field condition. The corresponding Green's function therefore must solve (11),

$$\Delta_x G(\mathbf{x};\mathbf{x}_0) = 0, \quad \mathbf{x} \neq \mathbf{x}_0, \quad \lim_{|\mathbf{x}|\to\infty} G(\mathbf{x},\mathbf{x}_0) = 0. \tag{17}$$

The normalization condition (12) is the same as (4), namely

$$\int_{\partial B} \nabla_x G(\mathbf{x};\mathbf{x}_0)\cdot\hat{\mathbf{n}} = 1. \tag{18}$$

for any ball B centered on x_0. Let us observe the following: if we rotate the Green's function about x_0, it still will solve (17-18) since the Laplace operator is invariant under rotation. Thus G only depends on the distance between $\mathbf{x}$ and $\mathbf{x}_0$. We can write $G = G(r), r = |\mathbf{x} - \mathbf{x}_0|$, where in spherical coordinates (17) is

$$\frac{1}{r^2}(r^2 G'(r))' = 0 \text{ if } r \neq 0, \quad \lim_{r\to\infty} G(r) = 0. \tag{19}$$

This is easily integrated twice to give the general solution

$$G = -\frac{c_1}{r} + c_2, \tag{20}$$

where $c_2 = 0$ by using the far-field condition in (19). The normalization condition (18) determines c_1. Letting B be the unit ball centered at x_0,

$$1 = \int_{\partial B} \nabla_x G(\mathbf{x}; \mathbf{x}_0) \cdot \hat{\mathbf{n}} = \int_{\partial B} c_1 \, dx^2 = 4\pi c_1,$$

so that $c_1 = 1/4\pi$. Thus the Green's function is $G(\mathbf{x}; x_0) = -1/(4\pi|\mathbf{x} - \mathbf{x}_0|)$, and the solution to (16) is

$$u(\mathbf{x}) = -\int_{\mathbb{R}^3} \frac{f(\mathbf{x}_0)}{4\pi|\mathbf{x} - \mathbf{x}_0|} dx_0^3.$$

Two dimensions. The two dimensional version is slightly different. In this case we want to solve

$$\Delta u = f, \quad \lim_{R\to\infty} \Big(\int_1^R u_r(r, \theta) \, dr - u(R, \theta) \Big) = 0. \tag{21}$$

The far field condition looks very strange at first glance. The reason for this is that solutions to $\Delta u = f$ will in general NOT decay to zero at large $|x|$, but will grow logarithmically. Again we look for a Green's function of the form $G = G(r)$, so that

$$\frac{1}{r}(rG'(r))' = 0 \text{ if } r \neq 0, \quad \lim_{R\to\infty} \Big(\int_1^R G_r \, dr - G \Big) = 0. \tag{22}$$

The general solution is

$$G = c_1 \ln r + c_2, \tag{23}$$

where $c_2 = 0$ by using the far-field condition in (22). The normalization condition (18) gives (where B is the unit disk)

$$1 = \int_{\partial B} \nabla_x G(\mathbf{x}; \mathbf{x}_0) \cdot \hat{\mathbf{n}} = \int_{\partial B} c_1 \, dx^2 = 2\pi c_1,$$

so that $c_1 = 1/2\pi$. Thus the Green's function is $G(\mathbf{x}; \mathbf{x}_0) = \ln|\mathbf{x} - \mathbf{x}_0|/2\pi$, and the solution to (21) is

$$u(\mathbf{x}) = \int_{\mathbb{R}^2} \frac{\ln|\mathbf{x} - \mathbf{x}_0| f(\mathbf{x}_0)}{2\pi} dx_0^2.$$

3 Inhomogeneous boundary conditions

Remarkably, the Green's function can be used for problems with inhomogeneous boundary conditions even though *the Green's function itself satisfies homogeneous boundary conditions*. We will use two ingredients to derive the appropriate representation: (1) a "Green's formula", particular to the linear operator in question and (2) the notion of reciprocity of the Green's function.

3.1 Green's formulas

The fact that $\mathcal{L}$ is self-adjoint with respect to the usual L^2 inner product means that for all functions u, v satisfying the homogeneous boundary conditions in problem (1),

$$\int_\Omega (\mathcal{L}v)u \, d\mathbf{x} - \int_\Omega (\mathcal{L}u)v \, d\mathbf{x} = 0. \tag{24}$$

What if u, v don't necessarily satisfy homogeneous boundary conditions? Then something like (24) would still be true, but terms involving boundary values of u, v would appear:

$$\int_\Omega (\mathcal{L}v)u \, d\mathbf{x} - \int_\Omega (\mathcal{L}u)v \, d\mathbf{x} = \text{boundary terms involving } u \text{ and } v. \tag{25}$$

What this formula actually looks like depends on the linear operator in question and is known as the *Green's formula* for the linear operator G.

3.2 Reciprocity (Symmetry) of the Green's function

A useful fact is that $G(\mathbf{x}; \mathbf{x}_0) = G(\mathbf{x}_0; \mathbf{x})$ provided G is self-adjoint so that equation (24) holds. Let $v(\mathbf{x}) = G(\mathbf{x}; \mathbf{x}_1)$ and $u(\mathbf{x}) = G(\mathbf{x}; \mathbf{x}_2)$. Then because of (8), we have $\mathcal{L}v = \delta(\mathbf{x} - \mathbf{x}_1)$ and $\mathcal{L}u = \delta(\mathbf{x} - \mathbf{x}_2)$. Plugging these into (24) gives

$$\int_\Omega \delta(\mathbf{x} - \mathbf{x}_1) G(\mathbf{x}; \mathbf{x}_2) d\mathbf{x} - \int_\Omega G(\mathbf{x}; \mathbf{x}_1) \delta(\mathbf{x} - \mathbf{x}_2) d\mathbf{x} = 0 \tag{26}$$

Using the basic property of the delta function, this simplifies to

$$G(\mathbf{x}_1;\mathbf{x}_2) - G(\mathbf{x}_2;\mathbf{x}_1) = 0 \tag{27}$$

which is what we wanted to show.

3.3 Green's formula representation for inhomogeneous boundary conditions

Let's now use (25) to solve the problem

$$\mathcal{L}u(\mathbf{x}) = f(\mathbf{x}) \quad + \text{inhomogeneous boundary conditions.} \tag{28}$$

We do this by setting $v(\mathbf{x}) = G(\mathbf{x};\mathbf{x}_0)$ in the green's formula (25), giving

$$\int_\Omega (\mathcal{L}G)u\,d\mathbf{x} - \int_\Omega (\mathcal{L}u)v\,d\mathbf{x} = \text{boundary terms involving } f, G. \tag{29}$$

Because $\mathcal{L}G = \delta(\mathbf{x}-\mathbf{x}_0)$ and $\mathcal{L}u = f$, we have

$$\int_\Omega \delta(\mathbf{x}-\mathbf{x}_0)u(\mathbf{x})d\mathbf{x} - \int_\Omega f(\mathbf{x})G(\mathbf{x};\mathbf{x}_0)d\mathbf{x} = \text{boundary terms involving } f, G \tag{30}$$

We can collapse the integral involving the δ function, and reverse the notation $\mathbf{x}$ and $\mathbf{x}_0$:

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}_0;\mathbf{x})f(\mathbf{x}_0)d\mathbf{x}_0 + \text{boundary terms involving } f, G \tag{31}$$

Note that this is not exactly like (6). Not only are there extra boundary terms, $\mathbf{x}$ and $\mathbf{x}_0$ are reversed in the function G. Reciprocity comes to the rescue, allowing us to write

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x};\mathbf{x}_0)f(\mathbf{x}_0)d\mathbf{x}_0 + \text{boundary terms involving } f, G \tag{32}$$

which is exactly the formula (6) with additional boundary terms.

3.4 Green's formula representation for the Laplace equation

The discussion above was meant to sketch the structure of the Green's function solution for a general linear equation with general inhomogeneous boundary conditions. Now let's see exactly what this looks like for the particular case $\mathcal{L} = \Delta$.

One dimension. In this case, the Green's formula (25) is nothing more than integration by parts twice (i.e. Green's second identity in 1-D), which for $\mathcal{L} = d^2/dx^2$ and $\Omega = [0, L]$ reads

$$\int_0^L uv'' - vu''dx = [uv' - vu']_0^L. \tag{33}$$

If we want to solve

$$u_{xx} = f, \quad u(0) = A, u(L) = B, \tag{34}$$

we again use the Green's function (15) appropriate for *homogeneous* boundary conditions. Plugging $v(x) = G(x; x_0)$ into (33), we get

$$\int_0^L u(x)G_{xx}(x; x_0) - G(x; x_0)u''(x)dx = [u(x)G_x(x; x_0) - G(x; x_0)u'(x)]_{x=0}^{x=L}. \tag{35}$$

Using the equation for u replacing G'' with the delta function,

$$u(x_0) = \int_0^L G(x; x_0)f(x)dx + [u(x)G_x(x; x_0) - G(x; x_0)u'(x)]_{x=0}^{x=L}. \tag{36}$$

Since $G = 0$ on the boundaries, this simplifies to

$$u(x) = \int_0^L G(x; x_0)f(x_0)dx_0 + BG_x(x; L) - AG_x(x; 0). \tag{37}$$

where we have interchanged x and x_0 and used reciprocity.

Many dimensions. In this case, the Green's formula (25) is nothing more than Green's second identity,

$$\int_\Omega u\Delta v - v\Delta u \, dx^n = \int_{\partial\Omega} u\nabla v \cdot \hat{\mathbf{n}} - v\nabla u \cdot \hat{\mathbf{n}} \, dx^{n-1}. \tag{38}$$

Suppose then we wish to solve the problem with the inhomogeneous boundary condition

$$\Delta u = f \text{ on } \Omega, \quad u = h \text{ on } \partial\Omega,$$

(Note that we're not specifying the dimension n of Ω, since the formulas will be valid in any dimension). Let G be the Green's function that solves $\Delta G = \delta(\mathbf{x} - \mathbf{x}_0)$ with homogeneous, Dirichlet boundary conditions. Then substituting $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_0)$ in (38), we have (being careful to keep $\mathbf{x}$ as the variable of integration)

$$\int_\Omega u(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_0)-G(\mathbf{x};\mathbf{x}_0)f(\mathbf{x})dx^n = \int_{\partial\Omega} u(\mathbf{x})\nabla_x G(\mathbf{x};\mathbf{x}_0)\cdot\hat{\mathbf{n}}-G(\mathbf{x};\mathbf{x}_0)\nabla u(\mathbf{x})\cdot\hat{\mathbf{n}}dx^{n-1}. \tag{39}$$

Using the definition of the δ-function and the fact that G is zero on the domain boundary, this simplifies to

$$u(\mathbf{x}_0) = \int_\Omega G(\mathbf{x};\mathbf{x}_0)f(\mathbf{x})dx^n + \int_{\partial\Omega} h(\mathbf{x})\nabla_x G(\mathbf{x};\mathbf{x}_0)\cdot\hat{\mathbf{n}}\,dx^{n-1}. \tag{40}$$

We can change our notation by reversing $\mathbf{x}$ and $\mathbf{x}_0$, and use reciprocity to produce a more standard looking result

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x};\mathbf{x}_0)f(\mathbf{x}_0)dx_0^n + \int_{\partial\Omega} h(\mathbf{x}_0)\nabla_{x_0} G(\mathbf{x};\mathbf{x}_0)\cdot\hat{\mathbf{n}}\,dx_0^{n-1}. \tag{41}$$

The first term is just the solution we expect for homogeneous boundary conditions. The second term is more surprising: it is the *derivative* of G that goes into the formula to account for the inhomogeneous Dirichlet boundary condition.

3.5 Green's formula representation using the free space Green's function

Now let's repeat the calculation of the previous section using the free space Green's function in $\mathbb{R}^3$ instead of the one particular to the domain Ω. Inserting $v(\mathbf{x}) = -1/(4\pi|\mathbf{x}-\mathbf{x}_0|)$ in (38) and going through the same operations, we arrive at

$$u(\mathbf{x}) = \int_\Omega f(\mathbf{x}_0)dx_0^n+\int_{\partial\Omega} u(\mathbf{x}_0)\nabla_{x_0}\left(-\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}\right)\cdot\hat{\mathbf{n}}+\left(\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}\right)\nabla u(\mathbf{x}_0)\cdot\hat{\mathbf{n}}dx_0^{n-1}. \tag{42}$$

If $f = 0$, this is exactly the representation formula 7.2.1 in Strauss's text. Notice that the right hand side uses *both* Dirichlet and Neumann data on the boundary. Usually both of these are not known, so (42) does not solve the problem directly. On the other hand, (42) is useful as a formula to *connect* the values of solutions and their normal derivatives on the boundary.

Exercise. Show

$$\nabla_{x_0}\left(-\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}\right) = -\frac{\mathbf{x}-\mathbf{x}_0}{4\pi|\mathbf{x}-\mathbf{x}_0|^3}.$$