1. Show that if $u, w$ solve two different, linear inhomogeneous equations $L u = g$ and $L w = h$, then $v = u + w$ solves $L v = g + h$. (This means we can divide the right hand side into separate terms and deal with each separately)

2A. Show that

$$\langle f, g \rangle = \int_{-1}^{1} x^2 f(x) g(x) \, dx$$

is an inner product on functions $f : [-1, 1] \to R$.

2B. For which positive integer exponents $m, n$ will $x^n, x^m$ be orthogonal (with respect to this inner product)?

3A. What is the adjoint of the differential operator (with respect to the $L^2$ inner product $\langle f, g \rangle = \int_0^1 f(x) g(x) \, dx$)

$$L = x \frac{d^2}{dx^2},$$

acting on $C^2_0[0, 1]$ (i.e. twice differentiable functions whose values are zero on the boundaries $x = 0, L$). Hint: write down $\langle L f(x), g(x) \rangle$ and move the derivatives off $f(x)$ using integration by parts. You will be left with $\langle f(x), L^1 g(x) \rangle$.

3B. Find a weight function $w(x)$ so that $L$ is self-adjoint with respect to the inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x) w(x) \, dx.$$

4A. Solve the Sturm-Liouville eigenvalue problem (i.e. find all eigenvalues and corresponding eigenfunctions)

$$u''(x) + \lambda u(x) = 0,$$

with boundary conditions $u(0) = 0, u'(L) = 0$. What is the difference between this problem and the one in the notes which has the boundary condition $u(L) = 0$?
4B. One can show that the linear operator $d^2/dx^2$ (acting on functions with the correct boundary conditions) is self-adjoint with respect to the usual $L^2$ inner product. Show that all the eigenfunctions you found are orthogonal with respect to the same inner product.

4C. Let $L = 1$. Write $f(x) = 2x - x^2$ as a linear combination (i.e. an expansion) of eigenfunctions.

5A. A liquid film of thickness $h(x,t)$ flows because of surface tension. The flux of $h$ is $h^3 h_{xxx}$. Assume $h$ is conserved and write a partial differential equation for $h$. Also assume $h > 0$.

5B. Suppose that the liquid cannot flow out of the spatial domain $x \in [0, 1]$. Express this fact as a boundary condition.

5C. Show that for any $A, B, C$, $h(x,t) = Cx^2 + Bx + A$ is a solution to the equation and boundary condition you derived (these solutions correspond to menisci of constant curvature).

5D. A quantity $F$ is called *dissipated* (as opposed to conserved) if $dF/dt \leq 0$. For the equation you have derived, show that

$$F = \int_0^1 h_x^2 dx$$

is dissipated, provided $h_{xxx} = h_x = 0$ on the boundaries. (Hint: move the time derivative inside the integral, integrate by parts in $x$, and then substitute in $u_t$. A final integration by parts will produce a non-positive answer)