Homework 10: Fourier transforms
Math 456/556

1. (practice with transforms)
A. Text problem 12.3.3
B. Text problem 12.3.4(a)
C. Text problem 12.3.5 (a)

2. (Solving an integral equation) Suppose we want to find a function $f(x)$ which solves the **integral** equation

$$\int_{-\infty}^{\infty} e^{-|x-y|} f(y)dy = g(x)$$

where $g(x)$ is known. Take a Fourier transform to solve for f. (Hint: the integral is a convolution. It should also be remarked that this is like a Green's function problem in reverse: you want to know the source term f given the solution g)

3. (like 12.4.6) Use the Fourier transform to solve $\Delta u = 0$ on an infinite strip $\{0 < y < 1, -\infty < x < \infty\}$ with boundary conditions $u(x,0) = 0$ and $u(x,1) = f(x)$ (take the transform in the x variable **only** to obtain an ODE in y only).

4. (Convective diffusion equation) Suppose that in addition to diffusion, there is something (i.e. wind or water currents) convecting a conserved quantity $u(x,t)$. The flux of material is the sum of diffusive and convective fluxes

$$J = -Du_x + Vu$$

Plugging into the general form for conservation laws gives the convection - diffusion equation

$$u_t = Du_{xx} - Vu_x, \quad -\infty < x < \infty.$$

Suppose that $u(x,0) = f(x)$ initially.
A. Solve this equation by Fourier transform (you'll need the translation property: the transform of $u(x-a)$ is $exp(-iax)$ times the transform of $u(x)$)
B. Make the change of variables $y = x - Vt$. Show that $u(y,t) = u(x - Vt, t)$ satisfies the usual diffusion equation. Therefore, the solution in part (a) is just the usual solution to the diffusion equation evaluated at $x - Vt$.

5. Consider the fourth order diffusion equation

$$u_t = -u_{xxxx}, \quad -\infty < x < \infty.$$

A. Take a Fourier transform to obtain an ODE in time.
B. Suppose one was looking for the *source function* $S(x, t)$ with initial condition $u(x, 0) = \delta(x)$. Find the **transform** of the source function by solving the ODE in part A using an appropriate initial condition.
C. The inverse transform of part B is NOT EASY! Suppose that $h(x)$ is a function whose Fourier transform is

$$H(k) = \exp(-k^4).$$

Use the dilation property for Fourier transforms (the transform of $af(ax)$ is $F(k/a)$) to write S in terms of h. (By the way, h has a graph that looks like a little like a Gaussian, except that it oscillates for large $|x|$)
D. Write the solution for a general initial condition $u(x, 0) = f(x)$ as an integral involving $h(x)$.