1. Let $u(x,t), 0 < x < L, t > 0$ be the complex valued solution of Schrödinger’s equation and boundary conditions

$$u_t = iu_{xx}, \quad u(0) = 0 = u(L).$$

(Note this looks like the diffusion equation with imaginary diffusivity). Using separation of variables, write the solution as a series given the initial condition $u(x,0) = g(x)$. Do solutions decay in time or oscillate?

2. Consider the “damped” wave equation and boundary conditions

$$u_{tt} = c^2 u_{xx} - ru_t, \quad u(0) = 0 = u(L).$$

Using separation of variables, write the solution as a series given the initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$. (Write your answer in real form - that is, don’t write exponentials that are potentially complex valued)

3. For the wave equation with Dirichlet boundary conditions $u(0,t) = 0 = u(L,t)$, we found the temporal frequencies

$$\omega_n = \frac{n\pi c}{L}, \quad n = 1, 2, 3, ....$$

when we separated variables, i.e. the time-dependent parts had the form $\sin(\omega t)$ or $\cos(\omega t)$. The lowest frequency $\omega_1$ is sometimes called the fundamental frequency (for sound waves, this is the pitch of the note you hear). The other temporal frequencies are called harmonics, which in this case are just multiples of the fundamental

$$\omega_n = n\omega_1, \quad n = 2, 3, 4, ...$$

(By the way, the presence of other harmonics defines the timbre or tone of the sound you hear. $n = 2$ is one octave above the fundamental, $n = 3$ is an octave and fifth above, etc.)
A. Now consider the boundary conditions $u_x(0, t) = 0 = u(L, t)$. By solving the appropriate eigenvalue problem, determine the (temporal) frequencies in this case, and determine what multiples of the fundamental frequency are present.

B. Write down the solution to the initial value problem if $L = \pi/2$, $c = 1$, $u(x, 0) = \cos(3x)$ and $u_t(x, 0) = \cos(7x)$.

4. Find a solution to Laplace’s equation and boundary conditions

$$u_{xx} + u_{yy} = 0, \quad u(x, 0) = x, \quad u(x, 1) = 0, \quad u_x(0, y) = 0, \quad u_x(1, y) = y^2.$$ 

(Hint: $u$ can be written as the sum of solutions of two problems, one which has homogeneous boundary conditions on the top and bottom, the other on the left and right. Note that ultimately you need to compute Fourier sine coefficients of $y^2$ and cosine coefficients of $x$.)

5. Solve Laplace’s equation on the quarter disk $0 \leq r < a$, $0 < \theta < \pi/2$, where $u(r, 0) = 0 = u(r, \pi/2)$ and $u_r(a, \theta) = \sin(2\theta) + \sin(6\theta)$. 