Homework 4: Green’s functions and distributions
Math 456/556

Question 1
Consider the mapping
\[ h[\phi] = \int_0^1 x\phi(x) \, dx, \quad \phi \in \mathcal{D}. \]

A. Show that \( h[\phi] \) is a distribution. If it is written in the form
\[ h[\phi] = \int_{-\infty}^{\infty} g(x)\phi(x) \, dx, \]
what is the function \( g(x) \)?

B. Find the distributional derivative of \( h[\phi] \) using the definition. Write your answer in the form
\[ \int_{-\infty}^{\infty} g(x)\phi(x) \, dx \]
where \( g(x) \) is the sum of regular and \( \delta \)-type functions.

Question 2
A simple model of forced mechanical oscillations is
\[ u''(t) + u(t) = f(t), \]
where \( u(t) \) is the displacement and \( f(t) \) is the external force. The choice \( f = \delta(t - t_0) \) physically represents an intense, brief impact.

A. Using the definition, find the second distributional derivative \( u''(t) \) of the function
\[ u(t) = \begin{cases} 0 & t < 0, \\ \sin(t) & t > 0, \end{cases} \]
and write it in the form
\[ \int_{-\infty}^{\infty} g(x)\phi(x) \, dx, \]
where \( g(x) \) is the sum of a regular function and a \( \delta \)-type function.

B. Show that \( u \) solves \( u'' + u = \delta(t) \) in the sense of distributions.

Question 3
Using the definition, find the distributional Laplacian (in \( \mathbb{R}^2 \)) of \( f(r, \theta) = \ln(r) \) in terms of \( \delta \)-type functions.

Question 4
Consider the problem
\[ \frac{d^2 u}{dx^2} = f(x), \quad u(0) = 0, \quad \frac{du}{dx}(L) = 0. \]

A. Find the Green’s function \( G(x; x_0) \) which is continuous at \( x = x_0 \), satisfies the jump condition
\[ \lim_{x \to x_0^+} G_x(x; x_0) - \lim_{x \to x_0^-} G_x(x; x_0) = 1, \]
and boundary conditions
\[ G_{xx}(x; x_0) = 0 \text{ when } x \neq x_0, \quad G(0; x_0) = 0, G_x(L; x_0) = 0. \]

B. Write the solution \( u \) in terms of the \( f \) and the Green’s function you found, and verify your formula for the source term \( f(x) = x^2 \).
Question 5 Find the Green's function for the following problem:

\[
\frac{d^2 u}{dx^2} - \frac{2}{x^2} u = f(x), \quad u(0) = 0, \quad \lim_{x \to \infty} u(x) = 0.
\]

(Hint: the homogeneous equation is of Euler type, so solutions are powers of \(x\)) Write down the solution \(u(x)\) in terms of the Green's function.