Homework 5  
Math 456/556

**Question 1** Consider the boundary value problem

\[ u_{xx} = f(x), \quad u(0) = A, \quad u_x(1) = B, \]

(a) Find a solution in terms of the Green’s function (see previous homework!), using the Green’s formula

\[ \int_0^1 uv'' - vu'' \, dx = [uv' - vu']_0^1. \]

Write your answer in terms of known functions rather than just the generic \( G(x; x_0) \) notation.

(b) Check your result with \( u(x) = 1 - x \) so that \( f \equiv 0 \).

**Question 2** Find a solution to

\[ u_{xx} - u = f(x), \quad u'(0) = A, \quad \lim_{x \to \infty} u(x) = 0, \]

in terms of the appropriate Green’s function, using the Green’s formula

\[ \int_0^\infty u(v'' - v) - v(u'' - u) \, dx = [uv' - vu']_0^\infty. \]

Recall that the free-space Green’s function was \( G_\infty = -\frac{1}{2} \exp(-|x - x_0|) \).

**Question 3** (The vortex patch) The so-called streamfunction \( \psi : \mathbb{R}^2 \to \mathbb{R} \) in fluid mechanics solves \( \Delta \psi = \omega \) where \( \omega \) is a measure of the fluid rotation (the velocity field is perpendicular to the gradient of \( \psi \), by the way). A simple model of a hurricane of radius \( R \) has

\[ \omega = \begin{cases} 1 & |x| < R, \\ 0 & |x| \geq R. \end{cases} \]

Use the Green’s function appropriate for two dimensions to express the solution \( \psi \) as an integral in **polar coordinates**. You will want to use the law of cosines

\[ |x - x_0|^2 = |x_0|^2 + |x|^2 - 2|x||x_0| \cos(\varphi) \]

where \( \varphi \) is the angle between vectors \( x \) and \( x_0 \). Finally, evaluate \( \psi(0) \) (you can actually evaluate the integral in general, but it’s a lot more complicated!).
Question 4 (The Helmholtz equation in 3D) We want the Green's function \( G(x; x_0) \) for the equation 
\[ \Delta u - k^2 u = f(x) \] in \( \mathbb{R}^3 \) and far field condition \( \lim_{|x| \to \infty} u(x) = 0 \), where \( k > 0 \) is some constant.

(a) Like the Laplace equation, we suppose that \( G \) only depends on the distance from \( x \) to \( x_0 \), that is \( G(x; x_0) = g(r) \) where \( r = |x - x_0| \). Going to spherical coordinates, we find \( g \) satisfies the ordinary differential equation
\[ \frac{1}{r^2} (r^2 g_r)_r - k^2 g = 0, \quad r > 0. \]

The trick to solving this is the change of variables \( g(r) = h(r)/r \). The equation for \( h(r) \) will have exponential solutions.

(b) The constant of integration in part (a) is found just like the Laplacian Green's function in \( \mathbb{R}^3 \), using the normalization condition
\[ \lim_{r \to 0} \int_{S_r(0)} \nabla_x G(x; 0) \cdot \hat{n} \, dx = 1, \]
where \( S_r(0) \) is a spherical surface of radius \( r \) centered at the origin. (Recall this comes from integrating the equation for \( G \) on the interior of \( S_r(0) \) and applying the divergence theorem).

(c) Finally, write down the solution to \( \Delta u - k^2 u = f(x) \) assuming \( u \to 0 \) as \( |x| \to \infty \). Write the answer explicitly as iterated integrals in Cartesian coordinates.

Question 5 (multiple images) Find the Green's function for \( \Delta u = f(x) \) for the following domains and boundary conditions. Express your answers in Cartesian coordinates.

(A) The first quadrant in \( \mathbb{R}^2 \), with boundary and far-field conditions

\[ u(x, 0) = 0, \quad u(0, y) = 0, \quad \lim_{|x| \to \infty} |\nabla u| = 0. \]

(Hint: locate image sources in each quadrant, choosing their signs appropriately so that the sum cancels on the \( x \)- and \( y \)-axes.)

(B) The quarter space in \( \mathbb{R}^3 \) given by \( x > 0, y > 0 \), with boundary and far-field conditions

\[ u_y(x, 0, z) = 0, \quad u_x(0, y, z) = 0, \quad \lim_{|x| \to \infty} u = 0. \]

Question 6 Consider Laplace's equation \( \Delta u = 0 \) in the upper half-plane.

(a) Find the appropriate Green's function which is subject to boundary conditions

\[ G(x, 0; x_0, y_0) = 0, \quad \lim_{y \to \infty} G(x, y; x_0, y_0) = 0 \]

(b) Use part (a) to write (in Cartesian coordinates) the solution to Laplace's equation subject to \( u(x, 0) = h(x) \).

(c) Use part (b) to find an explicit solution (not just an integral) if \( h(x) = H(x) \), the step function. (Hint: you may want to use \( \arctan(u) = \int 1/(1 + u^2) \, du \) where the chosen branch of \( \arctan(u) \) is such that \( \arctan(\pm \infty) = \pm \pi/2 \)). Check your answer!
Question 7  Consider the problem

$$\Delta u = f(r, \theta) \text{ inside a disk of radius } a,$$

with mixed boundary conditions

$$u(a, \theta) = h_1(\theta) \text{ for } 0 < \theta < \pi, \quad u_r(a, \theta) = h_2(\theta) \text{ for } \pi < \theta < 2\pi,$$

(a) What formal problem (equation and boundary conditions) does the Green’s function satisfy? (actually finding such a Green’s function is not trivial!)

(b) Suppose that the Green’s function, written in polar coordinates as $G(r, \theta; r_0, \theta_0)$ is known. Express the solution $u(r, \theta)$ in terms of $G$. Write integrals and functions explicitly in terms of polar coordinates.