**Homework 8: Green's functions, Part I**
**Math 456/556**

1. **One dimensional Green's functions.** A. In one dimension, the normalization condition is $\int_I \mathcal{L}G = 1$ where $I$ is any interval (a one dimensional ball) containing $x_0$. If $\mathcal{L} = d^2/dx^2$, show that this condition is equivalent to the "jump" condition on the derivative at $x_0$

$$\lim_{x \to x_0^+} G_x(x; x_0) - \lim_{x \to x_0^-} G_x(x; x_0) = 1.$$

In other words, $G$ is continuous but not differentiable at $x = x_0$.
B. Consider the 1D problem

$$\frac{d^2u}{dx^2} = f(x), \quad u(0) = 0, \quad \frac{du}{dx}(L) = 0.$$

Find the Green's function $G(x; x_0)$ which satisfies

$$G_{xx}(x; x_0) = 0 \text{ when } x \neq x_0, \quad G(0; x_0) = 0, G_x(L; x_0) = 0,$$

together with the jump condition.
C. Write the solution $u$ in terms of the $f$ and the Green's function you found. Verify your formula for the source term $f(x) = x^2$.

2. Let $G_\infty = 1/(4\pi|\mathbf{x} - \mathbf{x}_0|)$ be the Green's function that was derived for the domain $D = \mathbb{R}^3$. Suppose that $G(\mathbf{x}; \mathbf{x}_0)$ is the Green's function for $\Delta$ with Dirichlet conditions on a bounded, open connected domain $D$, that formally solves

$$\Delta_x G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0), \quad G(\mathbf{x}, \mathbf{x}_0) = 0 \text{ if } x \in \partial D.$$

This exercise shows that our definition of the Green's function for $\Delta$ on a bounded domain coincides with that in Strauss, section 7.3. In particular, we will shown part (iii) of his definition.
A. For each fixed $\mathbf{x}_0$, show the difference between the free-space and bounded domain Green's functions $H(x) = G_\infty(\mathbf{x}, \mathbf{x}_0) - G(\mathbf{x}, \mathbf{x}_0)$ solves Laplace's equation in all of $D$.
B. What boundary conditions on $\partial D$ does $H$ have? Explain why $H$ must be bounded, using the maximum principle. This shows that $G_\infty$ and $G$ basically have the same type of singularity at $\mathbf{x}_0$, even though they differ elsewhere by a bounded function.

3. Consider the problem

$$\Delta u = f \text{ inside a disk of radius } a,$$

with boundary conditions

$$u(a, \theta) = h_1(\theta) \text{ for } 0 < \theta < \pi, \quad u_r(a, \theta) = h_2(\theta) \text{ for } \pi < \theta < 2\pi,$$

Suppose that you know the Green's function $G(r, \theta; r_0, \theta_0)$ (recall it satisfies HOMOGENEOUS boundary conditions of the same mixed type!). Express the solution in terms of $G$.