Question 1  A. Find a dispersion relation of the form $\omega = \omega(k)$ for the elastic beam equation

$$u_{tt} = -u_{xxxx}, \quad -\infty < x < \infty.$$  

Show that the group velocity is twice the phase velocity.

B. Now add some friction to give

$$u_{tt} + \alpha u_t = -u_{xxxx}.$$  

Compute the dispersion relation of the form $\sigma = \sigma(k)$ instead, where $u = \exp(\sigma t + ikx)$. Classify the cases for different $\alpha$ as linearly stable, unstable, or marginally stable.

Question 2  Water waves are often observed to have crests (moving at the group velocity) traveling at half the speed of the ripples (moving at the phase velocity).

(a) Up to a constant multiple, what is the dispersion relation?

(b) “Reverse engineer” an equation of motion. (Hint: square the dispersion relation. What differential operators would produce the resulting powers of $\omega, k$ ?)

Question 3  Find the steady state solution $u_0(x)$ for the convection-diffusion equation

$$u_t = Du_{xx} - Vu_x, \quad D > 0.$$  

subject to the conditions $\lim_{x \to \infty} u(x, t) = 0$ and $u(0, t) = 5$. Does the equation have a steady state solution for all convection velocities $V$? (The resulting solution is an example of a boundary layer, which is set up when flow and diffusion counterbalance. This is only possible if the flow is toward the boundary at $x = 0$)

Question 4  Linearize the following equation about $u = 0$. Find a dispersion relation of the form $\omega = \omega(k)$ for the linear equation.

$$u_{xt} + [u^2]_{xx} - u_{xxxx} = u.$$  

Are the resulting waves dispersive? Answer this by finding the phase and group velocities.

Question 5  There are two constant-valued steady solutions of $u_t = u_{xx} + u^2 - 1$. Find them and linearize about each one. Which one is stable?

Question 6  The Cahn-Hilliard equation describes the separation of phases such as water and oil, by prescribing the evolution of the volume fraction $u(x, t)$ of one of the phases as

$$u_t = (g'(u) - u_{xx})_{xx}, \quad g(u) = u^2(1 - u)^2.$$  

Show that $u_0 \in (0, 1)$ is a constant-valued steady state solution, and it is unstable if $1/2 - \sqrt{3}/6 < u_0 < 1/2 + \sqrt{3}/6$ (the interpretation is as follows: if a mixture is dilute enough, the phases will not separate).