# First and second order linear wave equations

## 1 Simple first order equations

Perhaps the simplest of all partial differential equations is

$$u_t + cu_x = 0, \quad -\infty < x < \infty. \tag{1}$$

There are no boundary conditions required here, although to find a unique solution some kind of side condition is required. Equation (1) is sometimes called the transport equation, because it is the conservation law with the flux $cu$, where $c$ is the transport velocity.

We can view (1) as the directional derivative of $u$ in the direction $\mathbf{v} = (1, c)$ where $\mathbf{v}$ is a vector in $(t, x)$-space. Therefore (1) means that the function $u(x, t)$ is **constant** on each line parallel to $\mathbf{v}$. These lines have the equations $x - ct = x_0$ for any constant $x_0$, and are known as *characteristic curves*, or simply characteristics. Since $u$ is constant along such lines, it must be a function of $x - ct$ alone, and the most general solution of (1) is therefore

$$u(x, t) = f(x - ct). \tag{2}$$

If we were supplied with an initial condition to (1), we immediately find that $f(x) = u(x, 0)$. The solution (2) therefore merely translates the initial data at speed $c$ as time progresses. Note that the solution (2) can also be obtained by other means, including Fourier transforms.

More generally, equations of the form

$$au_x + bu_y = 0 \tag{3}$$

have the same structure as (1). In this case the characteristic curves are of the form $bx - ay = C$ for any constant $C \in \mathbb{R}$. It follows that all solutions in this case have the form

$$u(x, t) = f(bx - ay) \tag{4}$$

for some arbitrary function $f()$. Not every possible set of boundary conditions for (3) are compatible. This happens since $u(x, y)$ is constant along characteristic curves, and the boundary data must be the same for boundary points lying on the same characteristic.

**Example.** Suppose we wanted to solve

$$u_x = u_y, \quad u(x, y) : \mathbb{R}^2 \to \mathbb{R}, \quad u(x, y) = \sin(x) \text{ on the line } y = x.$$

By (4) the most general solution is $u(x, y) = f(x + y)$. On the line $y = x$ we have $\sin(x) = u(x, y) = f(2x)$. This is a type of *functional equation*, whereby an unknown function $f()$ is given implicitly. Setting $z = 2x$, we have $x = z/2$ and $f(z) = \sin(z/2)$. The complete solution is therefore $u(x, y) = \sin((x + y)/2)$.

## 2 Second order wave equations

Now consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0 \tag{5}$$

on the entire real line $x \in (-\infty, \infty)$. We can factor the linear operator to give

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0.$$

Setting $v = u_t + cu_x$, we get two first order equations

$$\begin{aligned} v_t - cv_x &= 0, \\ u_t + cu_x &= v. \end{aligned}$$

(this is like solving linear systems by factoring as $ABx = b$: we can introduce an intermediate quantity $w = Bx$, solve $Aw = b$ and finally solve $w = Bx$). The general solution to the first equation is just $v = h(x + ct)$ for some function $h$. Now we must solve

$$u_t + cu_x = h(x + ct). \tag{6}$$

This is an *inhomogeneous* equation, and we can attempt to solve it by looking for solutions of the form $u = u_{\text{hom}} + u_p$ where $u_p$ is a particular solution and $u_{\text{hom}}$ solves the homogeneous equation

$$u_t + cu_x = 0. \tag{7}$$

The general solution to (7) is $u_{\text{hom}} = g(x - ct)$. We guess a particular solution of the form $u_p = f(x + ct)$. Plugging it gives $f'(s) = h(s)/2c$, which means that in principle $f(s)$ can be found by integration. Therefore the whole solution $u = u_{\text{hom}} + u_p$ must have the general form

$$u = g(x - ct) + f(x + ct). \tag{8}$$

There are two sets of characteristic lines, one going forward at speed $c$ and the other going backward at speed $c$, along which the solution components $g()$ and $f()$ are constant, respectively.

Now we would like to satisfy the initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x)$$

Inserting into the general solution (8) gives

$$f(x) + g(x) = u_0(x), \tag{9}$$
$$f'(x) - g'(x) = \tfrac{1}{c} v_0(x). \tag{10}$$

Integrating the second of these gives

$$f(x) - g(x) = \frac{1}{c} \int_0^x v_0(x')dx' + K \tag{11}$$

where $K$ is some constant of integration (the lower bound on the integral was specified arbitrarily). We can now add and subtract (9) and (11) in order to find $f$ and $g$:

$$\begin{aligned} f(x) &= \tfrac{1}{2} \left( u_0(x) + \tfrac{1}{c} \textstyle\int_0^x v_0(x')dx' + K \right), \\ g(x) &= \tfrac{1}{2} \left( u_0(x) - \tfrac{1}{c} \textstyle\int_0^x v_0(x')dx' - K \right) \end{aligned}$$

Therefore the complete solution to the initial value problem is (notice that $K$ drops out)

$$u = f(x + ct) + g(x - ct) = \frac{1}{2} \left( u_0(x + ct) + u_0(x - ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} v_0(x')dx'. \tag{12}$$

This is known as d'Alembert's solution to the wave equation.

### 2.1 Other second order wave equations

The above method can be generalized to any second order PDE which can be factored as two transport equations. For example,

$$u_{xx} + (a-b)u_{xy} - abu_{yy} = 0 \tag{13}$$

can be factored as

$$\left(\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right)u = 0.$$

which can be written as the system

$$\begin{aligned} v_x + av_y &= 0, \\ u_x - bu_y &= v. \end{aligned}$$

Following the same logic as above, we see that the most general solution is

$$u(x,y) = f(y-ax) + g(y+bx)$$

for arbitrary functions $f, g$. In this case, the solution components each have their own "characteristic speed".

**Example.** We want a d'Alembert type solution for $u_{xx} + u_{xy} - 20u_{yy} = 0$ subject to initial conditions $u(x,0) = \phi(x)$ and $u_y(x,0) = \psi(x)$. Factoring gives

$$\left(\frac{\partial}{\partial x} + 5\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial y}\right)u = 0.$$

so that by the above discussion, a general solution can be written

$$u(x,y) = g(4x+y) + f(5x-y).$$

To satisfy the initial data, we need

$$g(4x) + f(5x) = \phi(x), \tag{14}$$

$$g'(4x) - f'(5x) = \psi(x). \tag{15}$$

Integration of the second equation gives

$$\frac{1}{4}g(4x) - \frac{1}{5}f(5x) = \int_0^x \psi(x')dx' + C. \tag{16}$$

Using elimination to solve the linear equations (14) and (16) gives

$$\begin{aligned} g(4x) &= \frac{4}{9}\phi(x) + \frac{20}{9}\int_0^x \psi(x')dx' + \frac{20C}{9}, \\ f(5x) &= \frac{5}{9}\phi(x) - \frac{20}{9}\int_0^x \psi(x')dx' - \frac{20C}{9}. \end{aligned}$$

It follows that

$$\begin{aligned} u(x,y) &= g(4x+y) + f(5x-y) = g(4(x+y/4)) + f(5(x-y/5)) \\ &= \frac{4}{9}\phi(x+y/4) + \frac{4}{9}\phi(x-y/5) + \frac{20}{9}\int_{x-y/5}^{x+y/4} \psi(x')dx'. \end{aligned}$$

### 2.2 Wave speed and domain of dependence

The formula (12) which solves (5) reveals that the solution at $(x, t)$ only depends on the initial condition on the interval $I_d(x, t) = (x - ct, x + ct)$. This is because waves in the second order wave equation travel both left and right with speed $c$, but no faster. As a consequence, initial data outside the interval $I_d$ cannot affect the solution at $(x, t)$ since it cannot travel fast enough along characersitic lines. The interval $I_d$ is known as the *domain of dependence*.

In contrast, consider the solution to the diffusion equation on a line

$$u(x, t) = \int_{-\infty}^{\infty} \frac{f(x_0)}{\sqrt{4\pi Dt}} e^{-(x-x_0)^2/(4Dt)} dx_0.$$

If the initial condition $f(x)$ is changed, the entire solution for $(x, t) \in \mathbb{R} \times \{t > 0\}$ will change, since the fundamental solution is never zero. The domain of dependence in this case is therefore the real line $I_d = \mathbb{R}$.