Fixed points, linearization and linearized dynamics in PDE models

Suppose we have a PDE which involves time of the form
\[ u_t = R(u, u_x, u_{xx}, \ldots) \]  
(1)

An equilibrium solution the dynamics is a function of \( x \) which solves
\[ R(u, u_x, u_{xx}, \ldots) = 0. \]  
(2)

This is itself a differential equation, although there is one fewer independent variables.

Frequently, the solution to (2) is just a constant \( u = u_0 \) in space as well as time. For example, for the diffusion equation with “Dirichlet” boundary conditions
\[ u_t = u_{xx}, \quad u(0, t) = 2 = u(1, t), \]  
(3)
it is easy to see that \( u(x, t) = 2 \) is a solution which does not depend on time or the space variable. In general, however, equilibria may depend on \( x \); for example, for the diffusion equation with mixed boundary conditions
\[ u_t = u_{xx}, \quad u(0, t) = 0, u_x(1, t) = 1, \]  
(4)
the equilibrium solution solves a two-point boundary value problem
\[ (u_0)_{xx} = 0, \quad u_0(0) = 0, (u_0)_x(1) = 1, \]  
(5)
whose solution is easily obtained as \( u_0 = x \).

Linearization. It is frequently useful to approximate a nonlinear equation with a linear one, since we know a lot more about linear equations. If \( u_0(x) \) is an equilibrium of an equation of the form (1), then we can look for solutions of the form
\[ u(x, t) = u_0(x) + \epsilon w(x, t) \]  
(6)
Plugging into (1) and keeping only the terms of order \( \epsilon \) always gives us a linear, time dependent equation for \( w \).

As an example, consider the “Fisher” equation with conditions at infinity
\[ u_t = u_{xx} + u(1 - u), \quad \lim_{x \to \pm \infty} u = 0. \]  
(7)
An equilibrium solution satisfies
\[ u_{xx} + u(1 - u) = 0, \quad \lim_{x \to \pm \infty} u = 0. \]  
(8)

1
Since we know $u = 0$ is an equilibrium for the logistic equation $u_t = u(1 - u)$, we can guess and check that it also works as a $x$-independent equilibrium solution for the PDE. Plugging (6) into (7) one gets
\[ \epsilon w_t = \epsilon w_{xx} + \epsilon w - \epsilon^2 w. \]
Keeping only terms of order $\epsilon$, we get the linearized Fisher equation
\[ w_t = w_{xx} + w. \]
(9)
This is a diffusion equation with a linear source term.

**Behavior of linear, constant coefficient equations.** We now specialize to the case where (1) the spatial domain is $(-\infty, \infty)$ and (2) the equation only involves terms which are linear in $u$ and derivatives with $x$-independent coefficients. It turns out such equations have solutions of the form
\[ w(x, t) = \exp(\sigma t + ikx), \]
(10)
where $k$ is real (we could have used $\sin(kx)$ and $\cos(kx)$ but the complex version is prettier). For each $k$, there is a corresponding $\sigma$ that can be found by plugging (10) into the PDE. This gives a functional relationship $\sigma(k)$ called the “dispersion” relation. If $\sigma$ always has negative real part, then solutions of the form (10) will always shrink exponentially. If the equation in question was the linearization about an equilibrium solution, we would call this equilibrium solution stable. On the other hand, if $\sigma$ was positive for some values of $k$, then there are at least some solutions that grow exponentially fast. This would correspond to linearized instability.

Consider the linearized Fisher equation (9). Substituting (10) into it, one gets
\[ \sigma \exp(\sigma t + ikx) = -k^2 \exp(\sigma t + ikx) + \exp(\sigma t + ikx), \]
or $\sigma = -k^2 + 1$. Since $\sigma > 0$ when $|k| < 1$, we would say that the equilibrium solution $u = 0$ is linearly unstable.