Homework 1 Solutions
Math 587

1) Find positive functions \( f(x), g(x) \), continuous on \((0, \infty)\) so that \( f = \mathcal{O}(g(x)) \) as \( x \to 0 \), but \( \lim_{x \to 0} f(x)/g(x) \) does not exist.

One can take \( f(x) = \sin(1/x) \) and \( g(x) = 1 \), for example.

2) Find all \( \alpha \) so that \( f = \mathcal{O}(\epsilon^\alpha) \) for the following:

(a) \( f(\epsilon) = \epsilon \tan(\epsilon) \)

(b) \( \ln(\ln(1/\epsilon)) \)

(a) Want to find all \( \alpha \) such that \( \lim_{\epsilon \to 0^+} \frac{\epsilon \tan(\epsilon)}{\epsilon^\alpha} < \infty \). Since \( \lim_{\epsilon \to 0^+} \epsilon \tan(\epsilon) = 0 \) then for any \( \alpha \leq 0 \) have \( f(\epsilon) = o(\epsilon^\alpha) \). Consider the case with \( \alpha > 0 \).

Taylor expanding \( \tan(x) \) about \( x = 0 \) have

\[
\tan(x) = x + \frac{1}{3}x^3 + O(x^5)
\]

so for \( \epsilon \) near 0 have

\[
\frac{\epsilon \tan(\epsilon)}{\epsilon^\alpha} = \epsilon^{2-\alpha} + O(\epsilon^{4-\alpha})
\]

Thus, as long as \( \alpha \leq 2 \) have \( \lim_{\epsilon \to 0^+} \frac{\epsilon \tan(\epsilon)}{\epsilon^\alpha} < \infty \) and so \( \epsilon \tan(\epsilon) = \mathcal{O}(\epsilon^\alpha) \) for \( \alpha \leq 2 \).

(b) Again want to find all \( \alpha \) satisfying \( \lim_{\epsilon \to 0^+} \frac{\ln(\ln(1/\epsilon))}{\epsilon^\alpha} < \infty \). First note that

\[
\lim_{\epsilon \to 0^+} \ln(\ln(1/\epsilon)) = \infty
\]

Thus, need \( \alpha < 0 \) otherwise \( \lim_{\epsilon \to 0^+} \frac{\ln(\ln(1/\epsilon))}{\epsilon^\alpha} = \infty \). Suppose \( \alpha < 0 \) then

\[
\lim_{\epsilon \to 0^+} \frac{\ln(\ln(1/\epsilon))}{\epsilon^\alpha} = \lim_{\epsilon \to 0^+} \frac{\ln(-\ln(\epsilon))}{\epsilon^\alpha}
\]

\[
= \lim_{\epsilon \to 0^+} \frac{\frac{1}{\epsilon \ln(\epsilon)}}{\alpha \epsilon^{\alpha-1}}
\]

\[
= \lim_{\epsilon \to 0^+} \frac{1}{\alpha \epsilon^{\alpha-1} \ln(\epsilon)}
\]

\[
= \lim_{\epsilon \to 0^+} \frac{\epsilon^{\lceil \alpha \rceil}}{\alpha \ln(\epsilon)} = 0
\]

So have \( \ln(\ln(1/\epsilon)) = \mathcal{O}(\epsilon^\alpha) \) for \( \alpha < 0 \).
3) Suppose that $f \sim a_1 \phi_1(\epsilon) + a_2 \phi_2(\epsilon) + \cdots + a_n \phi_n(\epsilon)$, where $a_k \neq 0$. Show this is equivalent to

$$f - \sum_{k=1}^{m-1} a_k \phi_k(\epsilon) \sim a_m \phi_m, \quad m = 1, \ldots, n$$

In practical terms, this means that the $m$-th term in an asymptotic expansion can be found as an approximation to $f$ minus the first $m-1$ terms in the approximation.

By definition

$$f - \sum_{k=1}^{m} a_k \phi_k(\epsilon) = o(\phi_m(\epsilon))$$

so that

$$f - \sum_{k=1}^{m-1} a_k \phi_k(\epsilon) = a_m \phi_m + o(\phi_m(\epsilon)) = o(\phi_{m-1}(\epsilon))$$

where the last follows since $\phi_k$ is well-ordered.

4) Find an infinite expansion of $\ln(1 + e^{-1})$ for $\epsilon \to 0$. (Hint: factor out the dominant term inside the log first)

$$\ln(1 + e^{-1}) = \ln(e^{1/\epsilon}(e^{-1/\epsilon} + 1))$$

$$= \frac{1}{\epsilon} + \ln(1 + e^{-1/\epsilon})$$

If $\epsilon << 1$ then $e^{-1/\epsilon}$ is very small so can Taylor expand $\ln(1 + x)$ about 0 to rewrite the log term in powers of $e^{-1/\epsilon}$. Continuing with the above have

$$\ln(1 + e^{1/\epsilon}) = \frac{1}{\epsilon} + \ln(1 + e^{-1/\epsilon})$$

$$= \frac{1}{\epsilon} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n/\epsilon}$$

5) Find the two-term asymptotic expansion for the solutions of $\cos x = x/\epsilon$.

Check your approximation for $\epsilon = 0.3$ against a numerical solution (obtained by, for example, plotting $\cos x$ against $x/\epsilon$).

As $\epsilon \to 0$ there is only one solution which is near 0. Trying the expansion

$$x = x_1 \epsilon + x_2 \epsilon^3 + O(\epsilon^5)$$
and plugging it into the Taylor expansion of $\cos(x)$ at $x = 0$ have

$$
\cos(x) = 1 - x^2/2 + O(x^4)
= 1 - \frac{1}{2}(x_1 \epsilon + x_2 \epsilon^3) + O(\epsilon^7)
= 1 - \frac{1}{2}x_1 \epsilon^2 - O(\epsilon^4)
= \frac{1}{\epsilon}(x_1 \epsilon + x_2 \epsilon^3)
= x_1 + x_2 \epsilon^2
$$

Matching powers of $\epsilon$ have $x_1 = 1, x_2 = -1/2$. So two term expansion is

$$
x \approx \epsilon - \epsilon^3/2
$$

At $\epsilon = 0.3$ the expansion has approximate solution $x = .2865$. The numerically obtained solution is $x = 0.28767$.

6) (Bender Orszag 7.5)

(a) Find expansion (at least to first order) for all roots of $\epsilon x^3 + x^2 - 2x + 1 = 0$.

(b) Consider $\epsilon x^8 - \epsilon^2 x^6 + x - 2 = 0$ Clearly there is one $O(1)$ root near $x = 2$. Looking for large roots which scale like $\epsilon^{-\alpha}$ (where $\alpha > 0$), show that there can be dominant balance between the first and last terms only, so the other root is $\sim (2/\epsilon)^{1/8}$.

(c) Optional: do part (c) of this exercise.

(a) When $\epsilon = 0$ then $x = 1$ is a double root. So for $\epsilon$ small expect two roots each $O(1)$. To find an appropriate expansion I tried $x = 1 + x_1 \epsilon^\beta$. Substituting this solution into the polynomial leads to the equation

$$
\epsilon + 3x_1 \epsilon^{\beta+1} + 3x_1^2 \epsilon^{2\beta+1}x_1^{\beta+1} + x_1^3 \epsilon^{3\beta+1} + 1 + 2x_1 \epsilon^{2\beta} + x_1^2 \epsilon^{2\beta} - 2 - 2x_1 \epsilon^{2\beta} + 1
= \epsilon + 3\epsilon^{\beta+1}x_1 + 3x_1^2 \epsilon^{2\beta+1} + x_1^3 \epsilon^{3\beta+1} + x_1^2 \epsilon^{2\beta} = 0
$$

Dominate balance can be obtained between the first and last term by setting $\beta = 1/2$. Then $x_1^2 = -1$ so have the two term approximation for the $O(1)$ roots as

$$
x = 1 \pm \epsilon^{1/2}
$$

To find leading order of the third root I tried $x = \epsilon$ as a solution which leads to

$$
x_0^3 \epsilon^{3\alpha+1} + x_1^2 \epsilon^{2\alpha} - 2x_1 \alpha + 1 = 0
$$
Can get dominate balance between last 3 terms if \( \alpha = 0 \) which leads to the \( O(1) \) roots. Can also achieve dominant balance if \( \alpha = -1 \) in which case have \( x_1 = -1 \). So to leading order the third root is

\[
x = -\epsilon^{-1}
\]

To get a two term approximate I continued the expansion in powers of \( \epsilon \) namely \( x = -\epsilon^{-1} + x_0 \). Substituting this into the polynomial and grouping powers of \( \epsilon \) shows that \( x_0 = -1 \). Thus the two term approximation of the third root is

\[
x = -\epsilon^{-1} - 1
\]

If \( \epsilon > 0 \) then the roots of \( O(1) \) are complex whereas for \( \epsilon < 0 \) the roots are real.

(b) Supposing large root scales to leading order like \( \epsilon^{-\alpha} \) with \( \alpha > 0 \) if we use the approximation \( x = x_0 \epsilon^{-\alpha} \) in the polynomial have

\[
x_1^8 \epsilon^{-8\alpha+1} - x_1^6 \epsilon^{-6\alpha+2} + x_1 \epsilon^{-\alpha} - 2 = 0
\]

Now want to check for the possibility of dominant balance between various terms

Case 1: First and Second Terms
Then \(-8\alpha + 1 = -6\alpha + 2 \) thus \( \alpha = -1/2 \). But want \( \alpha > 0 \) so roots are large. Thus, cannot get dominant balance in this case.

Case 2: First and Third Terms
Then \(-8\alpha + 1 = -\alpha \) so \( \alpha = 1/7 \). In this case the first and third terms are \( O(\epsilon^{-1/7}) \) and the other terms are \( O(1) \) thus can have dominant balance between the first and third terms.

Case 3: First and Last Terms
Here \( \alpha = 1/8 \) but in this case the first and last terms are \( O(1) \) but the third term is not \( O(\epsilon^{-1/8}) \) so don’t have dominant balance between first and last terms.

Case 4: Second and Third Terms
In this case need \( \alpha = 2/5 \) but then first terms is \( O(\epsilon^{-11/5}) \) while the second and third terms are \( O(\epsilon^{-2/5}) \).

Case 5: Second of Last Terms
Here \( \alpha = 1/3 \) but while th second and last terms are \( O(1) \) the first term is now \( O(\epsilon^{-5/3}) \) so no dominant balance here.

Case 6: Third and Last Terms
This case requires \( \alpha = 0 \) which will not find large roots.

As a result dominant balance for roots which scale like \( x = \epsilon^{-\alpha} \) only occurs between the first and third terms. This balance is obtained when \( \alpha = 1/7 \). In this case \( x_1 = -1 \) and so the large root \( \sim -1/\epsilon^{1/7} \).
7) Let $A, B$ be nonsingular $n \times n$ matrices. Find a two term expansion of $(A + \epsilon B)^{-1}$. (Hint: let $C$ be equal to inverse so that $C(A + \epsilon B) = I$)

Suppose $C$ is of the form $C = X_0 + \epsilon X_1 + O(\epsilon^2)$ where $X_i$ are $n \times n$ matrices. Substituting this into $C(A + \epsilon B) = I$ and matching powers of $\epsilon$ have

\[ I = C(A + \epsilon B) \\
= (X_0 + \epsilon X_1 + O(\epsilon^2))(A + \epsilon B) \\
= X_0 A + \epsilon (X_1 A + X_0 B) + O(\epsilon^2) \]

So have $X_0 A = I$ and $X_1 A + X_0 B = 0$ thus

\[
\begin{align*}
X_0 &= A^{-1} \quad (O(1)) \\
X_1 A &= -X_0 B \quad (O(\epsilon)) \\
X_1 &= -X_0 BA^{-1} \\
X_1 &= -A^{-1} BA^{-1}
\end{align*}
\]

And so $C = (A + \epsilon B)^{-1} = A^{-1} - \epsilon A^{-1} BA^{-1} + O(\epsilon^2)$