CRYSTALLINE ASPECTS OF GEOGRAPHY OF LOW DIMENSIONAL VARIETIES I: NUMEROLOGY

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1. Introduction

This is a modest attempt to study, in a systematic manner, the structure of low dimensional varieties using p-adic invariants. The main objects of interest in this paper are surfaces and threefolds. It is known that there are many (counter) examples of "pathological" or unexpected behavior in surfaces and even of threefolds. Our focus, instead, has been on, obtaining systematically, general positive results instead of counter examples. Here are some questions which this paper attempts to address.

It is has been known for some time that the famous inequality $c_1^2 \leq 3c_2$ (the Bogomolov-Miyaoka-Yau inequality) fails for surfaces in positive characteristic. From the point of view of this paper one should ask: is there a class of surfaces for which *some chern class inequality remains valid?* This of course is not a new question. It has already been proposed in [27]. However, as far as the author is aware, the crystalline or de Rham-Witt aspect of this problem has not been explored. This aspect of the problem was one of the main motivating factors in the present investigation. As an example of this point of view, we do provide a weaker inequality, (studied in the classical case in [4]) $c_1^2 \leq 5c_2 + 6b_1$ for a large class of surfaces which includes ordinary (more generally Hodge-Witt) surfaces and also surfaces which lift to W_2 with torsion free crystalline cohomology (that is Mazur-Ogus surfaces). We show in particular that the that the obstruction to proving the inequality $c_1^2 \leq 5c_2$ involves a de Rham-Witt torsion contribution as well as a contribution from the slopes of Frobenius. These investigations stem from certain invariants of non-classical nature, called Hodge-Witt numbers, which were introduced by T. Ekedahl (see [8]) and use a remarkable formula of R. Crew (see loc. cit.) and in particular the Hodge-Witt number $h_W^{1,1}$ of surfaces. Using Enriques' classification that the negativity of $h_W^{1,1}$ implies that the surface is either quasi-elliptic (so we are automatically in characteristic two or three) or the surface is of general type. Surfaces which do not satisfy $c_1^2 \leq 5c_2 + 6b_1$ are particularly extreme cases of failure of the Bogomolov-Miyaoka-Yau. They all exhibit de Rham-Witt torsion, non-degeneration of Hodge de Rham or presence of crystalline torsion. In particular Szpiro's examples (of surfaces which do not satisfy the Bogomolov-Miyaoka-Yau inequality) have highly non-degenerate slope spectral sequence (and either have crystalline torsion or non-degenerate Hodge de Rham spectral sequence).

The other question of interest, especially for surfaces is: how do the various p-adic invariants reflect in the Enriques'-Kodaira classification? Of course, the behavior of cohomological invariants is quite well-understood. But in positive characteristic

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there are other invariants which are of a non-classical nature. In fact our first task here is to expand our list of birational invariants. The new invariants of surfaces are at moment defined only for smooth surfaces. But they are of birational nature in the sense that: two smooth, projective surfaces which are birational surfaces have the same invariants. These are: the V-torsion, the Néron-Severi torsion, the exotic torsion and the domino associated to the only potentially non-trivial differential in slope spectral sequence of the surface; in particular the dimension of this domino, denoted here by $T^{0,2}$ is a birational invariant of smooth, projective surfaces. We also study torsion in $H^2_{cris}(X/W)$ in terms of the Enriques' classification, making precise several results found in the existing literature.

Applying our methods to Calabi-Yau threefolds, we obtain a complete characterization (in any positive characteristic) of Calabi-Yau threefolds with negative Hodge-Witt numbers. These are are precisely the threefolds for which the Betti number $b_3 = 0$. In particular such threefolds cannot lift to characteristic zero. All known examples of non-liftable Calabi-Yau threefolds have this property and it is quite likely that all non-liftable Calabi-Yau threefolds are of this sort.

Here is a more detailed plan of the paper. In Section 2 we recall results in crystalline cohomology, and the theory of de Rham Witt complex which we use in this paper. Reader familiar with [13], [16], [6], [7] and [8] is strongly advised to skip this section.

In Section 3 we begin with one of the main themes of the paper. We begin by proving (see Theorem 2.37.1) that the domino numbers of a smooth, projective variety whose Hodge de Rham spectral sequence degenerates and whose crystalline cohomology is torsion free are completely determined by the Hodge numbers and the slope numbers (in other words they are completely determined by the Hodge numbers and the slopes of Frobenius). This had been previously proved by Ekedahl [15] for abelian varieties. In Proposition 3.3.1 we note that for a smooth, projective surface $h^{1,1}_W$ is the only one which can be negative, the rest are non-negative. In Subsection 3.2 we use the Enriques classification of surfaces to prove that if the characteristic $p \geq 5$ and if $h^{1,1}_W$ is negative then X is of general type. In characteristics two and three there are exceptions to the assertions arising from certain quasi-elliptic surfaces. The result is proved by stepping through the classification and showing that for surfaces with Kodaira dimension at most one, $h^{1,1}_W$ is non-negative.

In Section 4 we take up the topic of Chern class inequalities of the type $c_1^2 \leq 5c_2$ and $c_1^2 \leq 5c_2 + 6b_1$ (such inequalities were considered in [4]). In Theorem 4.3.1 this inequality is established for a large class of surfaces. In particular the inequality is proved for any smooth, projective surface (with $p \geq 3$) which lifts to W_2 and whose crystalline cohomology is torsion free, surfaces which are ordinary (or more generally, Hodge-Witt). In Proposition 4.1.1 we record the fact that the inequality $c_1^2 \leq 5c_2$ is equivalent to the following inequality $m^{1,1} - T^{0,2} \leq b_1$, an inequality with terms involving slopes of Frobenius, de Rham-Witt contributions and Betti number b_1. In Proposition 4.4.3 we show that for surfaces which satisfy the inequality $m^{1,1} \geq 2p_g$, the inequality $c_1^2 \leq 5c_2$ holds. Here $m^{1,1}$ is the slope number (see 2.22 for the definition) and p_g is the geometric genus.

In Subsection 4.6 we investigate lower bounds on $h^{1,1}_W$. For instance we note that if X is of general type then $-c_1^2 \leq h^{1,1}_W \leq h^{1,1}$, except possibly for $p \leq 7$ and X is fibred over a curve of genus at least two and the generic fibre is a singular rational curve of genus at most four. We conjecture that if $b_1 \neq 0$ and $h^{1,1}_W < 0$ then $X \to \mathrm{Alb}(X)$

has one dimensional image (and so such surfaces admit a fibration with an irrational base).

In Section 5 we digress a little from our main themes. This section is may be well-known to the experts. We consider torsion in the second crystalline cohomology of X. It is well-known that torsion in the second cohomology of a smooth projective surface is a birational invariant, and in fact the following variant of this is true in positive characteristic (see Proposition 5.6.1): torsion of every species (i.e. Néron-Severi, the V-torsion, and the exotic torsion) is a birational invariant. We also note that surfaces of Kodaira dimension at most zero do not have exotic torsion. The section ends with a criterion for absence of exotic torsion which is often useful in practice. In Proposition 5.1.1 we prove that if X' and X are two smooth surfaces which are birational then they have the same domino number (in the case the only one of interest is $T^{0,2}$). Thus this is a new birational invariant of smooth surfaces which lives only in positive characteristic. In Theorem 5.9.3 we show that if X is a smooth, projective surface of general type with exotic torsion then X has Kodaira dimension $\kappa(X) \geq 1$. A surface with V-torsion must either have $\kappa(X) \geq 1$ or $\kappa(X) = 0$, $b_2 = 2$ and $p_g = 1$ or $p = 2$, $b_2 = 10$ and $p_g = 1$. This theorem is proved via Proposition 5.9.1 where we describe torsion in $H^2_{cris}(X/W)$ for surfaces with Kodaira dimension zero.

In Section 6 we digress again from our main theme to answer a question of Mehta (for surfaces). We show that any smooth, projective surface X of Kodaira dimension at most zero has a Galois étale cover $X' \to X$ such that $H^2_{cris}(X'/W)$ is torsion free. Thus crystalline torsion in these situations, can in some sense, be uniformized, or controlled. We do not know if this result should be true without the assumption on Kodaira dimension.

In Section 7 we take up the study of Hodge-Witt numbers of smooth projective threefolds. In Section 7.3 take up the study of Calabi-Yau threefolds with negative $h^{1,2}_W$ (which is the only one which can be negative). In Theorem 7.3.1 we give a characterization, valid in any characteristic, of Calabi-Yau threefolds with negative $h^{1,2}_W$. All such threefolds must have $b_3 = 0$. Such threefolds, of course, cannot be lifted to characteristic zero. In Proposition 7.4.1 we note that the Hirokado and Schröer threefolds are examples of smooth, projective, Calabi-Yau threefold for which $h^{1,2}_W < 0$.

In [17] which is a thematic sequel (but work in progress) to this paper we will study the properties of a refined Artin invariant of families of Mazur-Ogus surfaces and related stratifications.

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2. Notations and Preliminaries

2.1. Let p be a prime number and let k be a perfect field of characteristic p. Let $\overline{k}$ be an algebraic closure of k. Let $W = W(k)$ be the ring of Witt vectors of k and let $W_n = W/p^n$ be the ring of Witt vectors of k of length $n \geq 1$. Let K be the quotient field of W. Let σ be the Frobenius morphism $x \mapsto x^p$ of k and let $\sigma : W \to W$ be its canonical lift to W. We will also write $\sigma : K \to K$ for the extension of $\sigma : W \to W$ to K.

2.2. Following [16, page 90] we write R for the Cartier-Dieudonne-Raynaud algebra of k. Recall that R is a W-algebra generated by symbols F, V, d with the following relations:

$$\begin{aligned} FV &= p \\ VF &= p \\ Fa &= \sigma(a)F \qquad \forall\, a \in W \\ aV &= V\sigma^{-1} \qquad \forall\, a \in W \\ d^2 &= 0 \\ FdV &= d \\ da &= ad \qquad \forall\, a \in W \end{aligned}$$

The Raynaud algebra is graded $R = R^0 \oplus R^1$ where R^0 is the W-subalgebra generated by symbols F, V with relations above and R^1 is generated as an R^0 bi-module by d (see [16, page 90]).

2.3. Every element of R can be written uniquely as a sum

$$\sum_{n>0} a_{-n}V^n + \sum_{n\geq 0} a_n F^n + \sum_{n>0} b_{-n} dV^n + \sum_{n\geq 0} b_n F^n d \tag{2.3.1}$$

where $a_n, b_n \in W$ for all $n \in \mathbb{Z}$ (see [16, page 90]).

2.4. Any graded R-module M can be thought of as a complex $M = M^\bullet$ where M^i for $i \in \mathbb{Z}$ are R^0 modules and the differential $M^i \to M^{i+1}$ is given by d with $FdV = d$ (see [16, page 90]).

2.5. From now on we will assume that all R-modules are graded.

2.6. On any R-module M we define a filtration (see [16, page 92] by

$$\operatorname{Fil}^n M = V^n M + dV^n M \tag{2.6.1}$$

$$\operatorname{gr}^n M = \operatorname{Fil}^n M / \operatorname{Fil}^{n+1} M \tag{2.6.2}$$

In particular we set $R_n = R/\operatorname{Fil}^n R$.

2.7. We topologize an R module M by the linear topology given by $\operatorname{Fil}^n M$ (see [16, page 92]).

2.8. We write $\hat{M} = \operatorname{proj\,lim}_n M/\operatorname{Fil}^n M$ and call $\hat{M}$ the completion of M, and say M is complete if $\hat{M} = M$. Note that $\hat{M}$ is complete and one has $(\hat{M})^i = \operatorname{proj\,lim}_n M^i/\operatorname{Fil}^n M^i$ [16, section 1.3, page 90].

2.9. Let M be an R-module, for all $i \in \mathbb{Z}$ we let

$$Z^iM = \ker(d : M^i \to M^{i+1}) \tag{2.9.1}$$

$$B^iM = d(M^{i-1}) \tag{2.9.2}$$

The W-module Z^iM is stable by F but not by V in general and we let

$$V^{-\infty}Z^i = \bigcap_{r\geq 0} V^{-r}Z^i \tag{2.9.3}$$

where $V^{-r}Z^i = \{x \in M^i | V^r(x) \in Z^i\}$. Then $V^{-\infty}Z^i$ is the largest R^0 submodule of Z^i and $M^i/V^{-\infty}Z^i$ has no V-torsion (see [16, page 93]).

The W-module B^i is stable by V but not in general by F. We let

$$F^\infty B^i = \bigcup_{s\geq 0} F^sB^i.$$

Then $F^\infty B^i$ is the smallest R^0-submodule of M^i which contains B^i (see [16, page 93]).

2.10. The differential $M^{i-1} \to M^i$ factors canonically as

$$M^{i-1} \to M^{i-1}/V^{-\infty}Z^i \to F^\infty B^i \to M^i \tag{2.10.1}$$

(see [16, page 93, 1.4.5]) we write $\tilde{H}^i(M) = V^{-\infty}Z^i/F^\infty B^i$, is called the heart of the differential $d : M^i \to M^{i+1}$ and we will say that the differential is *heartless* if its heart is zero.

2.11. A (graded) R-module M is profinite if M is complete and for all n, i the W-module $M^i/\operatorname{Fil}^n M^i$ is of finite length [16, Definition 2.1, page 97].

2.12. A (graded) R-module M is coherent if it is of bounded degree, profinite and the hearts $\tilde{H}^i(M)$ are of finite type over W [16, Theorem 3.8, 3.9, page 118].

2.13. An R-module M is a domino if M is concentrated in two degrees (say) $0, 1$ and $V^{-\infty}M^0 = 0$ and $F^\infty B^1 = M^1$. If M is any R-module then the canonical factorization of $d : M^i \to M^{i+1}$, given in 2.10.1, gives a domino: $M^i/V^{-\infty}M^i \to F^\infty B^{i+1}M^i$ (see [16, 2.16, page 110]), which we call the *domino associated to* the differential $d : M^i \to M^{i+1}$.

2.14. Let M be a domino, then we define $T(M) = \dim_k M^0/VM^0$ and call it the dimension of the domino. If M is any R-module we write $T^i(M)$ for the dimension of the domino associated to the differential $M^i \to M^{i+1}$. It is standard that $T^i(M)$ is finite [16, Proposition 2.18, page 110].

2.15. Any domino M comes equipped with a finite decreasing filtration by R-submodules such that the graded pieces are certain standard one dimensional dominos U_j (see [16, Proposition 2.18, page 110]).

2.16. The U_j, one for each $j \in \mathbb{Z}$, provide a complete list of all the one dimensional dominos (see [16, Proposition 2.19, page 111]).

2.17. Further by [8, Lemma 4.2, page 12] one has

$$\mathrm{Hom}_R(U_i, U_j) = 0 \tag{2.17.1}$$

if $i > j$, and

$$\mathrm{Hom}_R(U_i, U_i) = k. \tag{2.17.2}$$

2.18. Let X be a scheme over k, we will write $H^*_{\text{cris}}(X/W)$ for the crystalline cohomology of X, whenever this exists. If X is smooth and proper, in [13], one finds the construction of the de Rham Witt complex $W\Omega^\bullet_X$. The construction of this complex is functorial in X. This complex computes $H^*_{cris}(X/W)$ and one has a spectral sequence (the *slope spectral sequence* of X)

$$E_1^{i,j} = H^j(X, W\Omega^i_X) \Rightarrow H^{i+j}_{cris}(X/W).$$

The construction slope spectral sequence is also functorial in X and for each $j \geq 0$, $H^j(X, W\Omega^\bullet_X)$ is a graded R-module. Modulo torsion, the slope spectral sequence always degenerates at E_1. We say that X is *Hodge-Witt* if the slope spectral sequence degenerates at E_1.

2.19. Let X/k be a smooth projective surface. Then recall the following standard notation for numerical invariants of X. We will write $b_i = \dim_{\mathbb{Q}_\ell} H^i_{et}(X, \mathbb{Q}_\ell)$, $q = \dim \mathrm{Alb}(X) = \dim \mathrm{Pic}^0(X)_{\text{red}}$; $2q = b_1$, $h^{ij} = \dim_k H^j(X, \Omega^i_X)$; and $p_g(X) = h^{0,2} = h^{2,0}$. Then one has the following form of the Noether's formula:

$$10 + 12p_g = K_X^2 + b_2 + 8q + 2(h^{0,1} - q), \tag{2.19.1}$$

where K_X is the canonical bundle of X and K_X^2 denotes the self intersection. The point is that all the terms are non-negative except possibly c_1^2. Further by [1, page 25] we have

$$0 \leq h^{0,1} - q \leq p_g. \tag{2.19.2}$$

Formula (2.19.1) is easily seen to be equivalent to the usual form of Noether's formula

$$12\chi(\mathcal{O}_X) = c_1^2 + c_2 = c_1^2 + \chi_{et}(X). \tag{2.19.3}$$

2.20. In addition, when the ground field $k = \mathbb{C}$, we get yet another form of Noether's formula which is a consequence of (2.19.3) and the Hodge decomposition:

$$h^{1,1} = 10\chi(\mathcal{O}_X) - c_1^2 + b_1. \tag{2.20.1}$$

We will call this the *Hodge-Noether formula.*

2.21. **Hodge-Witt invariants and other invariants.** In the next few subsections we recall results on Hodge-Witt numbers [8, page 85] of surfaces and threefolds. We recall the definition of Hodge-Witt numbers and their basic properties.

2.22. Let X be a smooth projective variety over a perfect field k. The slope numbers of X are defined by (see [8, page 85]):

$$\begin{aligned} m^{i,j} &= \sum_{\lambda \in [i-1, i)} (\lambda - i + 1) \dim_K H^{i+j}_{\text{cris}}(X/W)_{[\lambda]} \\ &\qquad + \sum_{\lambda \in [i, i+1)} (i + 1 - \lambda) \dim_K H^{i+j}_{\text{cris}}(X/W)_{[\lambda]}. \end{aligned}$$

where the summation is over all the slopes of Frobenius λ in the indicated intervals.

2.23. Let X be a smooth projective variety. Then the domino numbers $T^{i,j}$ of X are defined by (see [8, page 85]):

$$T^{i,j} = \dim_k \mathrm{Dom}^{i,j}(H^\bullet(X, W\Omega_X^\bullet))$$

in other words, $T^{i,j}$ is the dimension of the domino associated to the differential $d : H^j(X, W\Omega_X^i) \to H^j(X, W\Omega_X^{i+1})$.

2.24. Let X be a smooth projective variety over a perfect field k. The Hodge-Witt numbers of X are defined by the formula (see [8, page 85]):

$$h_W^{i,j} = m^{i,j} + T^{i,j} - 2T^{i-1,j+1} + T^{i-2,j+2}.$$

2.25. The Hodge-Witt numbers, domino numbers and the slope numbers satisfy the following properties which we will now list. See [8] for details.

2.25.1. For all i, j one has the symmetries (see [8, Lemma 3.1, page 112]):

$$m^{i,j} = m^{j,i} = m^{n-i,n-j},$$

the first is a consequence of Hard Lefschetz Theorem and the second is a consequence of Poincaré duality. Further these numbers are obviously non-negative:

$$m^{i,j} \geq 0.$$

2.25.2. The following formulae give relations to Betti numbers (see [8, Theorem 3.2, page 85]):

$$\sum_{i+j=n} m^{i,j} = \sum_{i+j=n} h_W^{i,j} = b_n$$

and one has Ekedahl's upper bound (see [8, Theorem 3.2, page 86]):

$$h_W^{i,j} \leq h^{i,j},$$

and if X is Mazur-Ogus (see Subsection 2.29 for the definition of a Mazur-Ogus variety) then we have

$$h_W^{i,j} = h^{i,j}.$$

2.25.3. One has the following fundamental duality relation for dominos due to Ekedahl (see [7, Corollary 3.5.1, page 226]): for all i, j, the domino $T^{i,j}$ is canonically dual to $T^{n-i-2,n-j+2}$ and in particular

$$T^{i,j} = T^{n-i-2,n-j+2}. \tag{2.25.1}$$

2.26. The Hodge and Hodge-Witt numbers of X satisfy a relation known as Crew's formula which we will use often in this paper. The formula is the following

$$\sum_j (-1)^j h_W^{i,j} = \chi(\Omega_X^i) = \sum_j (-1)^j h^{i,j}. \tag{2.26.1}$$

2.26.1. For any smooth projective variety of dimension at most three we have $h_W^{i,j} = h_W^{j,i}$ for all i, j (see [8, Corollary 3.3(iii), page 113]).

2.26.2. For any smooth projective variety of dimension n we have $h_W^{i,j} = h_W^{n-i,n-j}$ for all i, j (see [8, Corollary 3.2(i), page 113]).

2.27. For surfaces, the formulas in 2.26.1 take more explicit forms (see [8, page 85] and [14, page 64]). We recall them now as they will play a central role in our investigations. Let X/k be a smooth projective surface, K_X be its canonical divisor, T_X its tangent bundle, and let c_1^2, c_2 be the usual Chern invariants of X (so $c_i = c_i(T_X) = (-1)^i c_i(\Omega^1_X))$. Then the Hodge-Witt numbers of X are related to the other numerical invariants of X by means of the following formulae [8, page 114]:

$$h_W^{0,1} = h_W^{1,0} \tag{2.27.1}$$

$$h_W^{0,1} = b_1/2 \tag{2.27.2}$$

$$h_W^{0,2} = h_W^{2,0} \tag{2.27.3}$$

$$h_W^{0,2} = \chi(\mathcal{O}_X) - 1 + b_1/2 \tag{2.27.4}$$

$$h_W^{1,1} = b_1 + \frac{5}{6}c_2 - \frac{1}{6}c_1^2 \tag{2.27.5}$$

2.28. The formula for $h_W^{1,1}$ above and Noether's formula give the following variant of the Hodge-Noether formula of 2.20.1. We will call this variant the *Hodge-Witt-Noether* formula:

$$h_W^{1,1} = 10\chi(\mathcal{O}_X) - c_1^2 + b_1. \tag{2.28.1}$$

This formula will be central to our study of surfaces in this paper.

2.29. **Mazur-Ogus and Deligne-Illusie varieties.** In the next few subsections we enumerate the properties of a class of varieties known as Mazur-Ogus varieties. We will use this class of varieties at several different points in this paper as well as its thematic sequels so we elaborate some of the properties of this class of varieties here.

2.30. **Mazur-Ogus varieties.** A smooth, projective variety over a perfect field k is said to be a Mazur-Ogus variety if it satisfies the following conditions:

(1) The Hodge de Rham spectral sequence of X degenerates at E_1, and
(2) crystalline cohomology of X is torsion free.

2.31. **Deligne-Illusie varieties.** A smooth, projective variety over a perfect field k is said to be a Deligne-Illusie variety if it satisfies the following conditions:

(1) X admits a flat lifting to $W_2(k)$ and
(2) crystalline cohomology of X is torsion free.

2.32. **Remarks.**

(1) The class of Mazur-Ogus varieties is quite reasonable for many purposes and is rich enough to contain varieties with many de Rham-Witt torsion phenomena. For instance any K3 surface is Mazur-Ogus (in particular the supersingular K3 surface is Mazur-Ogus).
(2) We caution the reader that our definition of Deligne-Illusie varieties is more restrictive than that conceived by Deligne-Illusie. Nevertheless, we have the following restatement of [5].

Theorem 2.32.1. If $p > \dim(X)$ then any Deligne-Illusie variety X is a Mazur-Ogus variety.

2.33. **Remark.** It seems reasonable to expect that the inclusion of the class of Deligne-Illusie varieties in the class of Mazur-Ogus varieties is strict. However we do not know of an example. The class of Deligne-Illusie varieties is closed under products to this extent: if X, Y are Deligne-Illusie varieties and if $p > \dim(X) + \dim(Y)$ then $X \times_k Y$ is a Deligne-Illusie variety. We do not know if the class of Mazur-Ogus varieties is closed under products.

2.34. Recall from [25, Section 1.1, page 7] that thanks to the Cartier operator the Hodge de Rham spectral sequence

$$E_1 = H^q(X, \Omega^p_{X/k}) \implies H^{p+q}_{\mathrm{dR}}(X/k) \tag{2.34.1}$$

degenerates at E_1 if and only if the conjugate spectral sequence

$$E_2^{p,q} = H^p(X, \mathbb{H}^q_{\mathrm{dR}}(\Omega^\bullet_{X/k})) \implies H^{p+q}_{\mathrm{dR}}(X/k) \tag{2.34.2}$$

degenerates at E_2. The first spectral sequence, in any case, induces the Hodge filtration on the abutment while the second induces the conjugate Hodge filtration (see [20]). In particular we see that if X is Mazur-Ogus then both the Hodge de Rham and the conjugate spectral sequences degenerate at E_1 (and E_2 resp.). In any case $H^*_{dR}(X/k)$ comes equipped with two filtrations: the Hodge and the conjugate Hodge filtration.

2.35. Let X be a smooth, projective surface over an algebraically field k of characteristic $p > 0$. Then we have the exact sequence

$$0 \to B_1\Omega^1_X \to Z_1\Omega^1_X \to \Omega^1_X \to 0,$$

where the arrow $Z_1\Omega^1_X \to \Omega^1_X$ is the inverse Cartier operator. In particular we have the subspace $H^0(X, Z_1\Omega^1_X) \subset H^0(X, \Omega^1_X)$ which consists of closed global one forms on X. Using iterated Cartier operators (or their inverses), we get (see [13, Chapter 0, 2.2, page 519]) a sequence of sheaves $B_n\Omega^1_X \subset Z_n\Omega^1_X$ and the exact sequence

$$0 \to B_n\Omega^1_X \to Z_n\Omega^1_X \to \Omega^1_X \to 0,$$

and sequence of sheaves $Z_{n+1}\Omega^1_X \subset Z_n\Omega^1_X$. We will write

$$Z_\infty\Omega^1_X = \cap_{n=0}^\infty Z_n\Omega^1_X.$$

This is the sheaf of indefinitely closed one forms. We will say that a global one form is indefinitely closed if it lives in $H^0(X, Z_\infty\Omega^1_X) \subset H^0(X, \Omega^1_X)$. In general the inclusions $H^0(Z_\infty\Omega^1_X) \subset H^0(Z_1\Omega^1_X) \subset H^0(\Omega^1_X)$ may all be strict.

2.36. For a smooth, projective surface, the condition that X is Mazur-Ogus takes more tangible geometric forms which are often easier to check in practice. Part of our next result is implicit in [13]. We will use the class of Mazur-ogus surfaces in extensively in this paper as well as its sequel and in particular the following result will be frequently used.

Theorem 2.36.1. Let X be a smooth, projective surface over a perfect field k of characteristic $p > 0$. Consider the following assertions

(1) X is Mazur-Ogus
(2) $H^2_{cris}(X/W)$ is torsion free and $h^{1,1} = h^{1,1}_W$,
(3) $H^2_{cris}(X/W)$ is torsion free and every global 1-form on X is closed,
(4) $\mathrm{Pic}\,(X)$ is reduced and every global 1-form on X is indefinitely closed,

(5) the differentials $H^1(X,\mathcal{O}_X) \to H^1(X,\Omega^1_X)$ and $H^0(X,\Omega^1_X) \to H^0(X,\Omega^2_X)$ are zero,

(6) the Hodge de Rham spectral sequence

$$E_1^{p,q} = H^q(X,\Omega^p_{X/k}) \Rightarrow H^{p+q}_{dR}(X/k)$$

degenerates at E_1,

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Leftrightarrow$ (6).

Proof. It is clear from Ekedahl's Theorems (see 2.25.2) that (1) $\Rightarrow$ (2). So we prove (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). Consider the assertion (2) $\Rightarrow$ (3). By [13, Proposition 5.16, Page 632] the assumption of (2) that $H^2_{cris}(X/W)$ is torsion-free implies that $\mathrm{Pic}\,(X)$ is reduced, the equality

$$\dim H^0(Z_1\Omega^1_X) = \dim H^0(Z_\infty\Omega^1_X),$$

and also the equality

$$b_1 = h^1_{dR} = h^{0,1} + \dim H^0(Z_1\Omega^1_X).$$

Now our assertion will be proved using Crew's formula 2.26.1. We claim that the following equalities hold

$$h^{0,0}_W = h^{0,0} \quad (2.36.2)$$

$$h^{0,1}_W = h^{0,1} \quad (2.36.3)$$

$$h^{0,2}_W = h^{0,2} \quad (2.36.4)$$

$$h^{2,0}_W = h^{2,0} \quad (2.36.5)$$

$$h^{1,1} - h^{1,1}_W = 2(h^{1,0} - h^{0,1}) \quad (2.36.6)$$

The first of these is trivial as both the sides are equal to one for trivial reasons. The second follows from the explicit form of Crew's formula for surfaces (and the fact that $\mathrm{Pic}\,(X)$ is reduced). The third formula follows from Crew's formula and the first two computations as follows. Crew's formula 2.26.1 says that

$$h^{0,0}_W - h^{0,1}_W + h^{0,2}_W = h^{0,0} - h^{0,1} + h^{0,2}.$$

By the first two equalities we deduce that $h^{0,2}_W = h^{0,2}$. By Hodge-Witt symmetry and Serre duality we deduce that

$$h^{0,2}_W = h^{2,0}_W = h^{0,2} = h^{2,0}.$$

Again Crew's formula also gives

$$h^{1,0}_W - h^{1,1}_W + h^{1,2}_W = h^{1,0} - h^{1,1} + h^{1,2}$$

So we get on rearranging that

$$h^{1,1} - h^{1,1}_W = h^{1,0} - h^{1,0}_W + h^{1,2} - h^{1,2}_W.$$

By Serre duality $h^{1,2} = h^{1,0}$ and on the other hand

$$h^{1,2}_W = h^{1,0}_W = h^{0,1}_W = h^{0,1}$$

by duality Hodge-Witt symmetry (see 2.26.1, 2.26.2) and the last equality holds as $\mathrm{Pic}\,(X)$ is reduced. Thus we see that

$$h^{1,1} - h^{1,1}_W = 2(h^{1,0} - h^{0,1}).$$

Thus the hypothesis of (2) implies that

$$h^{0,1} = h^{0,1},$$

and as

$$h^{0,1} = \dim H^0(Z_1\Omega^1_X) = \dim H^0(Z_\infty\Omega^1_X).$$

So we have deduced that every global one form on X is closed and hence (2) $\Leftrightarrow$ (3) is proved.

Now (3) $\Rightarrow$ (1) is proved as follows. The only condition we need check is that the hypothesis of (3) imply that Hodge de Rham degenerates. The only non-trivial part of this assertion is that $H^1(\mathcal{O}_X) \to H^1(X, \Omega^1_X)$ is zero. By [13, Prop. 5.16, Page 632] we know that if $H^2_{cris}(X/W)$ is torsion free then the differential $H^1(\mathcal{O}_X) \to H^1(X, \Omega^1_X)$ is zero. Further by hypothesis of (3) we see that $H^0(X, \Omega^1_X) \to H^0(X, \Omega^2_X)$ also is zero. The other differentials in the Hodge de Rham spectral sequence are either zero for trivial reasons or are dual to one of the above two differentials and hence Hodge de Rham degenerates. So (3) implies (1).

Now let us prove that (3) $\Rightarrow$ (4). The first assertion is is trivial after [13, Prop. 5.16, Page 632]. Indeed the fact that $H^2_{cris}(X/W)$ is torsion free implies that $\mathrm{Pic}\,(X)$ is reduced and $H^0(X, Z_\infty\Omega^1_X) = H^0(X, Z_1\Omega^1_X)$ and by the hypothesis of (3) we have further that $H^0(X, Z_1\Omega^1_X) = H^0(X, \Omega^1_X)$. Thus we have deduced (3) $\Rightarrow$(4).

The remaining assertions are well-known and are implicit in [13, Prop. 5.16, Page 632] but we give a proof for completeness. Now assume (4) we want to prove (5). By the hypothesis of (4) and [13, Prop. 5.16, Page 632] we see that the differential $H^0(X, \Omega^1_X) \to H^0(X, \Omega^2_X)$ is zero. So we have to prove that the differential $H^1(\mathcal{O}_X) \to H^1(\Omega^1_X)$ is zero. We use the method of proof of [13, Prop. 5.16, Page 632] to do this. Let $f : X \to \mathrm{Alb}(X)$ be the Albanese morphism of X. Then we have a commutative diagram

(2.36.7)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(A, \Omega^1_A) & \longrightarrow & H^1_{dR}(A/k) & \longrightarrow & H^1(\mathcal{O}_A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(Z_1\Omega^1_X) & \longrightarrow & H^1_{dR}(X/k) & \longrightarrow & H^1(\mathcal{O}_X) & & \end{array}$$

with exact rows and the vertical arrows are injective and cokernel of the middle arrow is ${}_pH^2_{cris}(X/W)_{Tor}$ (the p-torsion of the torsion of $H^2_{cris}(X/W)$). Moreover the image of $H^1_{dR}(X/k) \to H^1(\mathcal{O}_X)$ is the $E^{0,1}_\infty$ term in the Hodge de Rham spectral sequence. Thus the hypothesis of (4) that $\mathrm{Pic}\,(X)$ is reduced implies that $H^1(\mathcal{O}_A) = H^1(\mathcal{O}_X)$, so we have $H^1(\mathcal{O}_A) = H^1(\mathcal{O}_X) \subset E^{0,1}_\infty = H^1(\mathcal{O}_X)$. Hence $E^{0,1}_\infty = H^1(X, \mathcal{O}_X)$, so that the differential $H^1(\mathcal{O}_X) \to H^1(\Omega^1_X)$ is zero. This proves (4) implies (5).

Now let us prove (5) $\Leftrightarrow$ (6). It is trivial that (6) implies (5). So we only have to prove (5) $\Rightarrow$ (6). This is elementary, but we give a proof. The differential $H^0(\mathcal{O}_X) \to H^0(\Omega^1_X)$ is trivially zero, so by duality $H^2(\Omega^1_X) \to H^2(\Omega^2_X)$ is zero. The differential $H^0(\Omega^1_X) \to H^0(\Omega^2_X)$ is zero by hypothesis of (5). This is dual to $H^2(\mathcal{O}_X) \to H^2(\Omega^1_X)$ hence which is also zero. The differential $H^1(\Omega^1_X) \to H^1(\Omega^2_X)$ is dual to the differential $H^1(\mathcal{O}_X) \to H^1(\Omega^1_X)$ which is zero by hypothesis of (5). Hence we have proved (5) $\Leftrightarrow$ (6). □

2.37. Let X be a smooth projective variety over a perfect field. The purpose here is to prove the following. This was proved for abelian varieties in [8].

Theorem 2.37.1. Let X be a smooth projective Mazur-Ogus variety over a perfect field. Then for all $i, j \geq 0$ the domino numbers $T^{i,j}$ are completely determined by the Hodge numbers of X and the slope numbers of X.

Proof. This proved by an inductive argument. The first step is to note that by the hypothesis and [8] one has

$$h^{i,j} = h_W^{i,j} = m^{i,j} + T^{i,j} - 2T^{i-1,j+1} + T^{i-2,j+2} \tag{2.37.2}$$

and so we get for all $j \geq 0$

$$T^{0,j} = h^{0,j} - m^{0,j} \tag{2.37.3}$$

so the assertion is true for $T^{0,j}$ for all $j \geq 0$. Next we prove the assertion for $T^{i,n}$ for all i. From the above equation we see that $T^{i,n} = h^{i,n} - m^{i,n} + 2T^{i-1,n+1} - T^{i-2,n+2}$ and the terms involving $n+1, n+2$ are zero. Now do a downward induction on j to prove the result for $T^{i,j}$: for each fixed j, the formula for $T^{i,j}$ involves $T^{i-1,j+1}$, $T^{i-2,j+2}$ and by induction hypothesis on j (for each i) these two domino numbers are completely determined by the Hodge and slope numbers. Thus the result follows. □

In particular as complete intersections in projective space are Mazur-Ogus, we have the following.

Corollary 2.37.4. Let X be a smooth projective complete intersection in projective space. Then $T^{i,j}$ are completely determined by the Hodge numbers of X and the slope numbers of X.

3. Enriques Classification and negativity of $h_W^{1,1}$

3.1. The main theorem we want to prove is Theorem 3.1.1. The proof of Theorem 3.1.1 is divided in to several parts and it uses the Enriques' classification of surfaces. We do not know how to prove the assertion without using Enriques' classification [1], [2].

Theorem 3.1.1. Let X/k be a smooth, projective surface over a perfect field k of characteristic $p > 0$. If $p \geq 5$ and $h_W^{1,1} < 0$ then X is of general type. If $p = 2, 3$ and $h_W^{1,1} < 0$ then X is either quasi-elliptic of Kodaira dimension one or of general type.

Remark 3.1.2. The restriction on the characteristic in Theorem 3.1.1 comes in because of quasi-elliptic surfaces (which exist in characteristic two and three). From [21, Corollary, page 480] and the formula for $h_W^{1,1}$ it is possible to write down examples of quasi-elliptic surfaces of Kodaira dimension one where this invariant is negative.

3.2. **Enriques' classification.** We briefly recall Enriques' classification of surfaces ([23], [2], [1]). Let X/k be a smooth projective surfaces. Then Enriques's classification is carried out by means of the Kodaira dimension $\kappa(x)$. All surfaces with $\kappa(X) = -\infty$ are ruled surfaces; the surfaces with $\kappa(X) = 0$ comprise of K3 surfaces, abelian surfaces, Enriques surfaces, non-classical Enriques surfaces (in characteristic two), bielliptic surfaces and non-classical hyperelliptic surfaces (in characteristic two and three). The surfaces with $\kappa(X) = 1$ are (properly) elliptic surfaces and finally the surfaces with $\kappa(X) = 2$ are surfaces of general type.

3.3. In this subsection we give two proofs of the following:

Proposition 3.3.1. Let X/k be a smooth projective surface. Then for $(i,j) \neq (1,1)$ we have $h_W^{i,j} \geq 0$.

First proof. The assertion is trivial for $h_W^{0,1} = h_W^{1,0} = b_1/2$. So we have to check it for $h_W^{2,0} = h_W^{0,2} = \chi(\mathcal{O}_X) - 1 + b_1/2$. Writing out this explicitly we have

$$h_W^{0,2} = h^{0,0} - h^{0,1} + h^{0,2} - 1 + b_1/2 \tag{3.3.2}$$

or as $h^{0,0} = 1$ (as X is connected) we get

$$h_W^{0,2} = h^{0,2} - (h^{0,1} - q) \tag{3.3.3}$$

where $q = b_1/2 = \dim_k \mathrm{Alb}(X)$ is the dimension of the Albanese variety of X. By [1, page 25] we know that $h^{0,1} - q \leq p_g = h^{0,2}$ and so the non-negativity assertion follows. □

Second Proof. This imitates the proof of Proposition 7.1.1. We use the definition of $h_W^{i,j} = m^{i,j} + T^{i,j} - 2T^{i-1,j+1} + T^{i-2,j+2}$. To prove the result it suffices to show that $T^{i-1,j+1}$ is zero for all $(i,j) \neq (1,1)$. This follows from the fact that in the slope spectral sequence of a smooth projective surface, there is at most one non-trivial differential (see [24], [13, Corollary 3.14, page 619]) and this gives vanishing of the domino numbers except possibly $T^{0,2}$, and if $(i-1, j+1) = (0,2)$ then $(i,j) = (1,1)$. □

3.4. **The case $\kappa(X) = -\infty$.** We begin by stepping through the Enriques' classification (see 3.2) and verifying the non-negativity of the Hodge-Witt number in all the cases.

Proposition 3.4.1. Let X/k be a smooth projective surface. If $\kappa(X) = -\infty$ then $h_W^{1,1} \geq 0$.

Proof. As $\kappa(X) = -\infty$, we know from [2] that either X is rational or it is ruled (irrational ruled). Assume X is irrational ruled. Then one has $c_1^2 = 8 - 8q$ and $\chi(\mathcal{O}_X) = 1 - q$. By Noether's formula $\chi(\mathcal{O}_X) = \frac{1}{12}(c_1^2 + c_2)$ we get

$$h_W^{1,1} = b_1 + \frac{5}{6}4(1-q) - \frac{1}{6}8(1-q) = b_1 + 2(1-q) = 2 \geq 0 \tag{3.4.2}$$

where we have used the fact that for a ruled surface $\mathrm{Pic}\,(X)$ is reduced (which follows from $p_g = 0$ for a ruled surface so $H^2(X, W(\mathcal{O}_X)) = 0$), and the fact that $b_1 = 2q$.

If X is rational, then either $X = \mathbb{P}^2$ or X is ruled, rational. In the first case $c_1^2 = 9$ and in the second case $c_1^2 = 8$. In both the cases $\chi(\mathcal{O}_X) = 1$ and we are done by an explicit calculation. □

3.5. Before proceeding further we record a lemma which allows us to reduce the question of $h_W^{1,1} < 0$ to minimal surfaces. The main point is to note that if under a blowup at a single point $h_W^{1,1}$ increases by one. This follows from properties of c_1^2, c_2 under blowups and the formula 2.27.1.

3.6. **Lemma.** If X is a smooth, projective surface with $h_W^{1,1}(X) < 0$, and if X has a minimal model X', then $h_W^{1,1}(X') < 0$.

3.7. The lemma follows from the observation recorded earlier that $h_W^{1,1}$ increases by one under blowups, so passing from X to X' involves a decrease in $h^{1,1}$. Thus we see that $h_W^{1,1}(X') < h_W^{1,1} < 0$.

3.8. **The** $\kappa(X) = 0$**.**

Proposition 3.8.1. Let X/k be a smooth projective, minimal surface with $\kappa(X) = 0$. Then $h_W^{1,1} \geq 0$.

Proof. This is easy: when $\kappa(X) = 0$, we know that $c_1^2 = 0$ and so it suffices to show that $\chi(\mathcal{O}_X) \geq 0$. This follows from the table in [1, page 25]. □

3.9. **The case** $\kappa(X) = 1$**.** The following proposition shows that $h_W^{1,1} \geq 0$ holds for surfaces of $\kappa(X) = 1$ unless the surface is quasi-elliptic. There are example of quasi-elliptic surfaces for which the result fails.

Proposition 3.9.1. Assume X is a smooth projective, minimal surface over a perfect field of characteristic $p > 0$ with $\kappa(X) = 1$. If $p = 2, 3$, assume that X is not quasi-elliptic. Then $h_W^{1,1} \geq 0$.

Proof. Under our hypothesis, X is a properly elliptic surface (i.e., the generic fibre is smooth curve of genus 1 and $c_1^2 = 0$. Hence it suffices to verify that $c_2 \geq 0$. As $c_2 = \chi_{et}(X)$, the required inequality is equivalent to proving $\chi_{et}(X) \geq 0$. This inequality is implicit in [1]; it can also be proved directly using the Euler characteristic formula (see [3][page 290, Proposition 5.1.6] and the paragraph preceding it). □

3.10. **Proof of 3.1.1.** Now we can assemble various components of the proof. Assume that X has $h_W^{1,1} < 0$. By Proposition 3.4.1 we have $h_W^{1,1} \geq 0$ for $\kappa(X) = -\infty$, so we may assume that $\kappa(X) \geq 0$. Then by Lemma 3.6 we may assume that X is already minimal. Now by Proposition 3.8.1 and Proposition 3.9.1 we have $h_W^{1,1} \geq 0$ if $0 \leq \kappa(X) \leq 1$. So $h_W^{1,1} < 0$ forces X to have $\kappa(X) > 1$ and so X is of general type.

4. Chern class inequalities

4.1. In this section we study the chern class inequality $c_1^2 \leq 5c_2$ and a weaker variant $c_1^2 \leq 5c_2 + 6b_1$. These were studied in [4]. It is, of course, well-known that $c_1^2 \leq 5c_2$ fails in some surfaces. The first observation we have, albeit an elementary one, is that the obstructions to proving $c_1^2 \leq 5c_2$ are of de Rham-Witt (i.e. involving torsion in the slope spectral sequence) and crystalline (i.e. involving slopes of Frobenius on $H^2_{cris}(X/W)$). This has not been noticed before.

Let us begin by recording some trivial but important consequences of the remarkable formula for $h_W^{1,1}$ (see 2.27.1). The main reason for writing them out explicitly is to illustrate the fact that obstructions to Chern class inequalities for surfaces are of crystalline (involving slope of Frobenius) and de Rham-Witt (involving the domino number $T^{0,2}$).

In what follows we will write $c_i = c_i(T_X)$, $b_1 = \dim H^1_{cris}(X/W) \otimes K$, $T^{0,2} = \dim \mathrm{Dom}^{0,2}(H^2(X, W(\mathcal{O}_X)) \to H^2(X, W\Omega^1_X))$.

Proposition 4.1.1. Let X be a smooth, projective surface over a perfect field of characteristic $p > 0$.

(1) Then the following conditions are equivalent:
   (a) the inequality $c_1^2 \leq 5c_2$ holds,

   (b) the inequality $h_W^{1,1} \geq b_1$ holds,
   (c) the inequality $T^{0,2} + b_1 \leq m^{1,1}$ holds.
(2) If X is Mazur-Ogus surface. Then the following are equivalent
   (a) the inequality $c_1^2 \leq 5c_2$ holds for X,
   (b) the inequality $h^{1,1} \geq b_1$ holds for X.
(3) If X is a Hodge-Witt surface. Then the following assertions are equivalent
   (a) the inequality $c_1^2 \leq 5c_2$ holds,
   (b) the inequality $m^{1,1} \geq 2m^{0,1}$ holds.

Proof. All the assertions are trivial consequences of the formulae

$$h_W^{1,1} = m^{1,1} - 2T^{0,2} \tag{4.1.2}$$

$$h_W^{1,1} = \frac{5c_2 - c_1^2}{6} + b_1 \tag{4.1.3}$$

$$b_1 = m^{0,1} + m^{1,0} \tag{4.1.4}$$

$$m^{0,1} = m^{1,0}. \tag{4.1.5}$$

and are left to the reader. □

4.2. **Consequences of $h_W^{1,1} \geq 0$.** The also record the main reason for our interest in $h_W^{1,1} \geq 0$.

Proposition 4.2.1. Let X/k be a smooth projective surface over a perfect field k. Then

$$h_W^{1,1} \geq 0$$

holds if and only if the inequality:

$$c_1^2 \leq 5c_2 + 6b_1 \leq 5c_2 + 12h^{0,1}. \tag{4.2.2}$$

holds. On the other hand if $h^{1,1} < 0$, then

$$c_1^2 \geq 5c_2. \tag{4.2.3}$$

Proof. The assertions follow easily from Ekedahl's formula (2.27.1) for $h_W^{1,1}$:

$$h_W^{1,1} = b_1 + \frac{5}{6}c_2 - \frac{1}{6}c_1^2 \tag{4.2.4}$$

Hence we see that $h_W^{1,1} \geq 0$ gives $h_W^{1,1} > 0$ gives $5c_2 - c_1^2 > -6b_1$ or $c_1^2 < 5c_2 + 6b_1$. Further we see that $h_W^{1,1} < 0$ implies that

$$b_1 + \frac{5}{6}c_2 - \frac{1}{6}c_1^2 < 0 \tag{4.2.5}$$

As $b_1 \geq 0$ the term on the left is not less than $\frac{5}{6}c_2 - \frac{1}{6}c_1^2$ and so

$$\frac{5}{6}c_2 - \frac{1}{6}c_1^2 \leq h_W^{1,1} < 0, \tag{4.2.6}$$

and the result follows. □

Remark 4.2.7. Let X is a smooth projective surface of general type. Clearly when $h_W^{1,1} < 0$ the Bogomolov-Miyaoka-Yau inequality also fails. And secondly if X satisfies $c_1^2 \leq 3c_2$ then $h_W^{1,1} \geq 0$. Thus the point of view which seems to emerge from the results of this section is that surfaces with $h_W^{1,1} < 0$ are somewhat more exotic than the ones for which $h_W^{1,1} \geq 0$. Indeed as was pointed out in [8], $h_W^{1,1}$ is a deformation invariant so surfaces with $h_W^{1,1} < 0$ do not even admit deformations which lift to characteristic zero.

Corollary 4.2.8. If X is a smooth projective surface for which (4.2.2) fails to hold, then the slope spectral sequence of X has infinite torsion and does not degenerate at E_1.

Proof. Indeed, this follows from the formula

$$h_W^{1,1} = m^{1,1} - 2T^{0,2}, \tag{4.2.9}$$

is just the definition of $h_W^{1,1}$. The claim now follows as $m^{1,1} \geq 0$ and hence $h_W^{1,1} < 0$ implies that $T^{0,2} \geq 1$. ☐

Remark 4.2.10. Thus we see that the counter examples to Bogomolov-Miyaoka-Yau inequality given in [30] are not Hodge-Witt.

4.3. **Surfaces for which $h_W^{1,1} \geq 0$ holds.** Our next result provides a large class of surfaces for which $h_W^{1,1} \geq 0$ does hold.

Theorem 4.3.1. Let X/k be a smooth, projective surface over a perfect field of characteristic $p > 0$. Assume X satisfies any one of the following hypothesis:

1. the surface X is Hodge-Witt,
2. or X is ordinary (in the sense of Bloch-Kato),
3. or X is a Mazur-Ogus surface,
4. or assume $p \geq 3$ and X is a Deligne-Illusie variety,
5. or assume $p = 2$, and X lifts to W_2.

Then X satisfies (4.2.2).

Proof. The assertion that (4.2.2) holds is equivalent to $h_W^{1,1} \geq 0$. Thus it suffices to prove that $h_W^{1,1} \geq 0$. But this follows from the fact that $T^{0,2} = 0$ as X is Hodge-Witt and $m^{1,1} \geq 0$ by definition. This follows from the third (via [5]) so it suffices to prove the fourth. We can simply invoke [8, Corollary 3.3.1, Page 86] which gives us $h_W^{1,1} = h^{1,1}$. However we give an elementary proof in the spirit of this paper. We will use the formulas $\chi(\mathcal{O}_X) = 1 - h^{0,1} + h^{0,2}$ and $c_2 = \chi_{et}(X) = 1 - b_1 + b_2 - b_3 + b_4 = 2 - 2b_1 + b_2$. By 2.19.1 we have

$$12\chi(\mathcal{O}_X) = c_1^2 + c_2, \tag{4.3.2}$$

or equivalently $c_1^2 = 12\chi(\mathcal{O}_X) - c_2$. Now the assertion would follow if we prove that $c_1^2 \leq 5c_2 + 6b_1$. But

$$\begin{aligned}
5c_2 - c_1^2 + 6b_1 &= 5c_2 - (12\chi(\mathcal{O}_X) - c_2) + b_1 && (4.3.3)\\
&= 6c_2 - 12\chi(O_X) + 6b_1 && (4.3.4)\\
&= 6(2 - 2b_1 + b_2) - 12\chi(\mathcal{O}_X) + 6b_1 && (4.3.5)\\
&= 12 - 12b_1 + 6b_2 - 12(1 - h^{0,1} + h^{0,2}) + 6b_1 && (4.3.6)\\
&= 6b_1 - 12h^{0,1} + 6b_2 - 12h^{0,2} && (4.3.7)
\end{aligned}$$

Thus we see that $5c_2 - c_1^2 + 6b_1 = 6(b_1 - 2h^{0,1}) + 6(b_2 - 2h^{0,2})$. By [5] and the hypothesis that the crystalline cohomology of X is torsion free we have $b_2 = h^{0,2} + h^{1,1} + h^{2,0}$. Or equivalently by Serre duality we get $b_2 = 2h^{0,2} + h^{1,1}$ and again by the hypothesis that te crystalline cohomology of X is torsion free we see that Pic (X) is reduced and so $b_1 = 2h^{0,1}$. Thus $5c_2 - c_1^2 + 6b_1 = 6h^{1,1}$ and so is non-negative and in particular we have deduced that $h^{1,1}_W = h^{1,1}$. The fifth assertion follows falls into two cases: assume X is not ruled, then this follows from [27] as the hypothesis imply that $c_1^2 \leq 3c_2$. If X is ruled one deduces this from our earlier result on surfaces with Kodaira dimension $-\infty$. □

Remark 4.3.8. For this remark assume that the characteristic $p \geq 3$. In the absence of crystalline torsion, $h^{1,1}_W$ detects obstruction to lifting to W_2. More precisely, if X has torsion free $H^2_{cris}(X/W)$, and $h^{1,1}_W < 0$, then X does not lift to W_2.

4.4. **Examples of Szpiro, Ekedahl.** As was pointed out in [8] the counter examples constructed by Szpiro in [30] also provide examples of surfaces which are beyond the (4.2.2) faultline. We briefly recall these examples. In [30] Szpiro constructed examples of smooth projective surfaces S together with a smooth, projective and non-isotrivial fibration $f : S \to C$ where the fibres has genus $g \geq 2$ and C has genus $q \geq 2$. Let $f_n : S_n \to C$ be the fibre product of f with the n^{th}-iterate of Frobenius $F_{C/k} : C \to C$. Then

$$c_2(S_n) = 4(g-1)(q-1) \tag{4.4.1}$$

$$c_1^2(S_n) = p^n d + 8(g-1)(q-1) \tag{4.4.2}$$

where $d = deg(f_*(\Omega^1_{X/C}))$ is a positive integer. Thus in this case, as was pointed in [8], $h^{1,1}_W \to -\infty$ as $n \to \infty$.

Proposition 4.4.3. Let X be a smooth, projective surface of general type with $\chi(\mathcal{O}_X) \geq 1$. Further assume that X is Hodge-Witt. If $m^{1,1} \geq 2p_g$ then $c_1^2 \leq 5c_2$ holds for X.

Proof. By the formula for $h^{1,1}_W$ we have

$$6h^{1,1}_W = 6(m^{1,1} - 2T^{0,2}) = 5c_2 - c_1^2 + 6b_1.$$

As X is Hodge-Witt we see that $T^{0,2} = 0$ and so

$$6(m^{1,1} - b_1) = 5c_2 - c_1^2.$$

Hence the asserted inequality holds if $m^{1,1} - b_1 \geq 0$. Writing

$$m^{1,1} - b_1 = (m^{1,1} - 2p_g) + (2p_g - 2q),$$

where we have used $b_1 = 2q$. Thus to prove the proposition it will suffice to prove that each of the two terms in the parenthesis are non-negative. The first holds by the hypothesis of the proposition and for the second, we see that $2p_g - 2q \geq 2p_g - 2h^{0,1} = 2(\chi(\mathcal{O}_X) - 1)$. Thus the proposition follows as we have $\chi(\mathcal{O}_X) \geq 1$ by our hypothesis. □

Remark 4.4.4. The assumption $\chi(\mathcal{O}_X) \geq 1$ is not too unreasonable. It was shown in [28] that if $p \geq 7$, then $\chi(\mathcal{O}_X) \geq 0$ for any smooth, projective surface of general type. Moreover if $\chi(\mathcal{O}_X) = 0$, then $c_2 < 0$ by Noether's formula 2.19.1, so by [27], X is uniruled (and in any case if $c_2 < 0$, then the inequality $c_1^2 \leq 5c_2$ is false).

4.5. **Weak Bogomolov-Miyaoka holds in characteristic zero.** Assume for this remark that $k = \mathbb{C}$, and that X is a smooth, projective surface. Then using the Hodge decomposition for X, Noether's formula can be written as

$$h^{1,1} = 10\chi(\mathcal{O}_X) - c_1^2 + b_1, \tag{4.5.1}$$

and as the left hand-side of this formula is always non-negative we deduce that $c_1^2 \leq 10\chi(\mathcal{O}_X) + b_1$. This is easily seen to be equivalent to $c_1^2 \leq 5c_2 + 6b_1$.

Remark 4.5.2. If the weak Bogomolov-Miyaoka-Yau (4.2.2) fails to hold for a smooth projective surface X of general type, then Ω^1_X is Bogomolov unstable. Indeed this follows from [27, Corollary 15]. To see that the conditions of that corollary are valid it suffices to verify that $c_1^2 > \frac{16p^2}{(4p^2-1)}c_2$. By hypothesis, (4.2.2) fails, so one has

$$c_1^2 > 5c_2 + 6b_1 \geq 5c_2 \tag{4.5.3}$$

and as

$$4 < \frac{16x^2}{(4x^2-1)} \leq 4.26\cdots \tag{4.5.4}$$

for $x \geq 2$. Thus the assertion follows from Shepherd-Barron's result.

4.6. **Lower bounds on $h^{1,1}_W$.** In this subsection we are interested in lower bounds for $h^{1,1}_W$. It turns out that unless we are in characteristic $p \leq 7$, the situation is not too bad thanks to a conjecture of Raynaud (which is a theorem of Shepherd-Barron).

Proposition 4.6.1. Let X be a smooth projective surface of general type. Then

(1) except when $p \leq 7$ and X is fibred over a curve of genus at least two and the generic fibre is a singular rational curve of arithmetic genus at most four we have

$$-c_1^2 \leq h^{1,1}_W \leq h^{1,1}. \tag{4.6.2}$$

(2) If $c_2 > 0$ then $h^{1,1}_W > -\frac{1}{6}c_1^2$.

(3) If X is not uniruled then

$$-\frac{1}{6}c_1^2 \leq h^{1,1}_W \leq h^{1,1}. \tag{4.6.3}$$

(4) If $h^{1,1}_W < -\frac{1}{6}c_1^2$ then there exists a morphism $X \to C$ with connected fibres and C has genus at least one.

Proof. We prove(1). Assume if possible that $h^{1,1}_W < -c_1^2$. Then by using the formula $h^{1,1}_W = b_1 + 10\chi(\mathcal{O}_X) - c_1^2$ we get $b_1 + 10\chi(\mathcal{O}_X) < 0$. As $b_1 \geq 0$ this implies that $\chi(\mathcal{O}_X) < 0$. By [27, Theorem 8] we know that any surface of general type with negative $\chi(\mathcal{O}_X)$ we have $p \leq 7$; and whenever $\chi(\mathcal{O}_X) < 0$ the surface X is fibred over a curve of genus at least two and the generic fibre is singular rational curve of genus at most four. Next we prove (2) and (3) which are really consequence of Raynaud's conjecture which was proved in [27]. From the formula for $h^{1,1}_W$ in terms of c_1^2, c_2, b_1. So suppose that X is not uniruled and assume, if possible, that

$$h^{1,1}_W < -\frac{1}{6}c_1^2 \tag{4.6.4}$$

Then writing out Ekedahl's formula for $h^{1,1}_W$ we get

$$h^{1,1}_W = b_1 + \frac{5}{6}c_2 - \frac{1}{6}c_1^2 < -\frac{1}{6}c_1^2, \tag{4.6.5}$$

and so this forces:

$$b_1 + \frac{5}{6}c_2 < 0 \tag{4.6.6}$$

and as $b_1 \geq 0$ we see that c_2 is negative. Now by [27, Theorem 7, page 263] we see that X is uniruled which contradicts our hypothesis. Now we prove (4). This is a part of the proof of Raynaud's conjecture in [27]. It is clear that the hypothesis implies that $c_2 < 0$. So by loc. cit. We know that the map $X \to \mathrm{Alb}(X)$ has one dimensional image, and this finishes the proof. □

Remark 4.6.7. (1) By a result of [21], exceptions in Theorem 4.6.1(1) do occur.
(2) Thus the examples of surfaces given in Subsection 4.4 satisfy the inequality in Theorem 4.6.1.

Conjecture 4.6.8. If X is a smooth projective surface with that $-\frac{1}{6}c_1^2 \leq h_W^{1,1} < 0$ and $b_1 \neq 0$ then the image of the Albanese map $X \to \mathrm{Alb}(X)$ is one dimensional.

5. Enriques classification and Torsion in crystalline cohomology

The main aim of this section is to explore geographical aspects of torsion in crystalline cohomology. It is well-known that if $X/\mathbb{C}$ is a smooth, projective surface then the torsion in $H^2(X, \mathbb{Z})$ is invariant under blowups. We will see a refined version of this result holds in positive characteristic (see Theorem 5.6.1). Our next result also provides a new birational invariant of smooth surfaces.

5.1. $T^{0,2}$ is a birational invariant.

Proposition 5.1.1. Let X, X' be smooth, projective surfaces and suppose that $X' \to X$ is a birational morphism. Then $T^{0,2}(X) = T^{0,2}(X')$.

Proof. Using the fact that any birational morphism $X' \to X$ of surfaces factors as finite sequence of blowups at closed points, we reduce to proving this assertion for the case when $X' \to X$ is the blowup at one closed point.

As c_2 increases by 1 and c_1^2 decreases by 1 under blowups, the formula for $h_W^{1,1}$ shows that $h_W^{1,1}(X') = h_W^{1,1}(X)+1$ while using the formula for blowups for crystalline cohomology and a slope computation shows that the slope numbers of X' and X satisfy

$$m^{1,1}(X') = m^{1,1}(X) + 1,$$

here the "1" is the contribution coming from the cohomology in degree two of the exceptional divisor which is one dimensional, so the result follows as

$$h_W^{1,1} = m^{1,1} - 2T^{0,2}.$$

□

5.2. In the next few subsections we will use the formulas which describe the behavior of cohomology of the de Rham-Witt complex under blowups. We recall these from [10]. Let X be a smooth projective variety and let $Y \subset X$ be a closed subscheme, pure of codimension d. Let X' denote the blowup of X along Y, and let $f : X' \to X$ be the blowing up morphism. Then one has

$$H^j(X, W\Omega_X^i) \oplus_{0<n<d} H^{j-n}(Y, W\Omega_Y^{i-n}) \xrightarrow{\simeq} H^j(X', W\Omega_{X'}^i). \tag{5.2.1}$$

5.3. **Birational invariance of the domino of a surface.** The de Rham-Witt cohomology of a surface has only one, possibly non-trivial, domino. This is the domino associated to the differential $H^2(X, W(\mathcal{O}_X)) \to H^2(X, W\Omega_X^1)$. In this section we prove the following.

Theorem 5.3.1. Let X, X' be two smooth, projective surfaces over an algebraically closed field k of characteristic $p > 0$ and let $X' \to X$ be a birational morphism. Then the domino associated to the differential $H^2(X, W(\mathcal{O}_X)) \to H^2(X, W\Omega_X^1)$ are isomorphic.

5.4. **Proof of 5.3.1.** As any birational morphism $X' \to X$ as above factors as a finite sequence of blowups at closed points, we may assume that $X' \to X$ is the blowup of X at a single point. In what follows, to simplify notation, we will denote objects on X' by simply writing them as primed quantities and the unprimed ones will denote objects on X. We will use the notation of Subsection 2.9.

The construction of the de Rham-Witt complex $W\Omega_X^\bullet$ is functorial in X. The properties of the de Rham-Witt complex (in the derived category of complexes of sheaves of modules over the Cartier-Dieudonne-Raynaud algebra) under blowing up have been studied extensively in [10], and using [10, Chapter 7, Theorem 1.1.9], and the usual formalism of de Rham-Witt cohomology, we also have a morphism of slope spectral sequences. The blowup isomorphisms described in the blowup formula fit into the following diagram

$$\begin{array}{ccccc} H^2(W(\mathcal{O}_{X'})) & \xrightarrow{d'} & H^2(W\Omega^1_{X'}) & \xrightarrow{d'} & H^2(W\Omega^2_{X'}) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(W(\mathcal{O}_{X})) & \xrightarrow{d} & H^2(W\Omega^1_{X}) & \xrightarrow{d} & H^2(W\Omega^2_{X}). \end{array}$$

By the blowup formula 5.2.1 all the vertical arrows are isomorphisms. This induces an isomorphism $Z' = \ker(d') \to \ker(d) = Z$.

Now the formula for blowup for cohomology of the de Rham-Witt complex also shows that we have isomorphisms for $i = 1, 2$,

$$H^i(X', W(\mathcal{O}_{X'})) \xrightarrow{\simeq} H^i(X', W(\mathcal{O}_X))$$

and these fit into the following commutative diagram.

$$\begin{array}{ccccccccc} H^1(W(\mathcal{O}_{X'})) & \longrightarrow & H^1(X', \mathcal{O}_{X'}) & \longrightarrow & H^2(W(\mathcal{O}_{X'})) & \longrightarrow & H^2(W(\mathcal{O}_{X'})) & \longrightarrow & H^2(\mathcal{O}_{X'}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(W(\mathcal{O}_{X})) & \longrightarrow & H^1(X, \mathcal{O}_{X}) & \longrightarrow & H^2(W(\mathcal{O}_{X})) & \longrightarrow & H^2(W(\mathcal{O}_{X})) & \longrightarrow & H^2(\mathcal{O}_{X}) \end{array}$$

In the above commutative diagram we claim that all the vertical arrows are isomorphisms. Indeed [11, Proposition 3.4, V.5] shows that $H^i(X', \mathcal{O}_{X'}) \to H^i(X, \mathcal{O}_X)$ are isomorphisms for $i \geq 0$. The other vertical arrows are isomorphisms by [10]. Thus from the diagram we deduce an induced isomorphism

$$\ker(H^2(W(\mathcal{O}_{X'})) \xrightarrow{V} H^2(W(\mathcal{O}_{X'}))) \xrightarrow{\simeq} \ker(H^2(W(\mathcal{O}_{X})) \xrightarrow{V} H^2(W(\mathcal{O}_{X}))) .$$

Thus the V-torsion in $H^2(W(\mathcal{O}_X))$ of X and X' in $H^2(W(\mathcal{O}_{X'}))$ are isomorphic (we will use this in the proof of our next theorem as well).

Now these two arguments combined also give the corresponding assertions for the composite maps dV^n (resp. $d'V'^n$). Thus we also have from a similar commutative diagram (with dV^n etc.) from which we deduce that we have isomorphisms $\ker(d'V'^n) = V'^{-n}Z' \to V^{-n}Z = \ker(dV^n)$. Thus we have an isomorphism of the intersection of

$$V'^{-\infty}Z' = \cap_n V'^{-n}Z' \to V'^{-\infty}Z' = \cap_n V^{-n}Z.$$

Thus we have in particular, isomorphisms

$$\frac{H^2(X', W(\mathcal{O}_{X'}))}{V'^{-n}Z'} \simeq \frac{H^2(X', W(\mathcal{O}_{X}))}{V^{-n}Z}.$$

Now in the canonical factorization of d (resp. d') in terms of their dominos we have a commutative diagram

$$\begin{array}{ccccccc} H^2(W(\mathcal{O}_{X'})) & \longrightarrow & \frac{H^2(W(\mathcal{O}_{X'}))}{V'^{-\infty}Z'} & \longrightarrow & F'^{\infty}B' & \longrightarrow & H^2(W\Omega^1_{X'}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^2(W(\mathcal{O}_X)) & \longrightarrow & \frac{H^2(W(\mathcal{O}_X))}{V^{-\infty}Z} & \longrightarrow & F^{\infty}B & \longrightarrow & H^2(W\Omega^1_X). \end{array}$$

The first two vertical arrows and the last are isomorphisms. Hence so is the remaining arrow. This completes the proof of the theorem.

5.5. **Crystalline Torsion.** We begin by quickly recalling Illusie's results about crystalline torsion. By crystalline torsion we will mean torsion in the W-module $H^2_{cris}(X/W)$, which we will denote by $H^2_{cris}(X/W)_{\mathrm{Tor}}$. Let X/k be a smooth projective variety. According to [13], torsion in $H^{cris}(X/W)$ arises from several different sources (see [13, Section 6]). Torsion in the Neron-Severi group of X, denoted $NS(X/k)_{\mathrm{Tor}}$ in this paper, injects into $H^2_{\mathrm{cris}}(X/W)$ via the crystalline cycle class map (see [13, Proposition 6.8, page 643]). The next species of torsion one finds in the crystalline cohomology of a surface is the V-torsion, denoted by $H^2_{cris}(X/W)_v$. It is the inverse image of V-torsion in $H^2(X, W(\mathcal{O}_X))$, denoted here by $H^2(X, W(\mathcal{O}_X))_{\mathrm{V-tors}}$, under the map $H^2(X/W) \to H^2(X, W(\mathcal{O}_X))$. It is disjoint from the Neron-Severi torsion (see [13, Proposition 6.6, page 642]). Torsion of these two species is collectively called the *divisorial torsion* in [13, page 643] and denoted by $H^2_{\mathrm{cris}}(X/W)_d$. The quotient

$$H^2_{\mathrm{cris}}(X/W)_e = H^2_{\mathrm{cris}}(X/W)_{\mathrm{Tor}}/H^2_{\mathrm{cris}}(X/W)_d$$

is called the *exotic torsion* of $H^2_{\mathrm{cris}}(X/W)$, or if X is a surface then simply by the exotic torsion of X.

5.6. Our next result concerns the torsion in the second crystalline cohomology of a surface.

Theorem 5.6.1. Let $X' \to X$ be a birational morphism of smooth projective surfaces. Then

(1) we have an isomorphism
$$H^2_{\mathrm{cris}}(X/W)_{Tor} \to H^2_{cris}(X'/W)_{Tor},$$
(2) and this isomorphism induces an isomorphism on the Neron-Severi, the V-torsion, and the exotic torsion.

Proof. As every $X' \to X$ as in the hypothesis factors as a finite sequence of blowups at closed points, it suffices to prove the assertion for the blowup at one closed point. So let $X' \to X$ be the blowup of X at one closed point $x \in X$. The formula for blowup for crystalline cohomology induces an isomorphism

$$H^2_{cris}(X/W)_{Tor} \xrightarrow{\simeq} H^2(X'/W)_{\mathrm{Tor}}.$$

This proves assertion (1). As remarked earlier, the proof of Theorem 5.3.1, also shows that the V-torsion of $H^2(W(\mathcal{O}_X))$ and $H^2(W(\mathcal{O}_{X'}))$ are isomorphic. Then by [13, Proposition 6.6, Page 642] we see that the V-torsion of X and X' are isomorphic. Thus we have an isomorphism on the V-torsion $H^2_{cris}(X/W)_v \simeq H^2_{cris}(X'/W)_v$.

Further it is standard that the Néron-Severi group of X does not acquire any torsion under blowup $X' \to X$. So we have an isomorphism

$$H^2_{cris}(X/W)_d \to H^2_{cris}(X'/W)_d,$$

of the divisorial torsion of X and X'. Therefore in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2_{cris}(X/W)_d & \longrightarrow & H^2_{cris}(X/W)_{Tor} & \longrightarrow & H^2_{cris}(X/W)_e & \longrightarrow & 0 \\ & & \downarrow \simeq & & \simeq \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^2_{cris}(X'/W)_d & \longrightarrow & H^2_{cris}(X'/W)_{Tor} & \longrightarrow & H^2_{cris}(X'/W)_e & \longrightarrow & 0 \end{array}$$

the first two columns are isomorphisms and the rows are exact so that the last arrow is an isomorphism. □

Remark 5.6.2. It is clear from Proposition 5.6.1 that while studying torsion in the crystalline cohomology of a surface that we can replace X by its smooth minimal model (when it exits).

5.7. The next result we want to prove is probably well-known to the experts. But we will prove a more precise form of this result in Theorem 5.9.3 and Proposition 5.9.1. We begin by stating the result in its coarsest form.

Theorem 5.7.1. *Let X/k be a smooth projective surface over a perfect field. If $\kappa(X) \leq 0$ then $H^2_{\rm cris}(X/W)$ does not have exotic torsion.*

5.8. **The case $\kappa(X) = -\infty$.** The case $\kappa(X) = -\infty$ is the easiest of all. If $\kappa(X) = -\infty$, then X is rational or ruled. If X is rational, by the birational invariance of torsion we reduce to the case $X = \mathbb{P}^2$ or X is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and in these case on deduces the result following result by inspection. Thus one has to deal with the case that X is ruled.

Proposition 5.8.1. *Let X be a smooth ruled surface over k. Then $H^2_{\rm cris}(X/W)$ is torsion free and X is Hodge-Witt.*

Proof. The first assertion follows from the formula for crystalline cohomology of a projective bundle over a smooth projective scheme. The second assertion follows from the following lemma which is of independent interest and will be of frequent use to us. □

Lemma 5.8.2. *Let X be a smooth, projective variety over a perfect field k.*

(1) *If $H^i(X, \mathcal{O}_X) = 0$ then $H^i(X, W(\mathcal{O}_X)) = 0$.*
(2) *If X/k is a surface with $p_g(X) = 0$ then X is Hodge-Witt.*

Proof. This is well-known and was also was noted in [19]. We include it here for completeness. Clearly, it is sufficient to prove the first assertion. We have the exact sequence

$$0 \to W_{n-1}(\mathcal{O}_X) \to W_n(\mathcal{O}_X) \to \mathcal{O}_X \to 0 \tag{5.8.3}$$

The result now follows by induction on n and the fact that $H^i(X, \mathcal{O}_X) = 0$. □

Lemma 5.8.4. *Let X be a smooth projective variety over a perfect field. If $H^2(X, \mathcal{O}_X) = 0$ then there is no exotic or V-torsion in $H^2_{\rm cris}(X/W)$.*

Proof. By Illusie's description of exotic torsion (see [13]) one knows that it is the quotient of a part of p-torsion in $H^2(X, W(\mathcal{O}_X))$, but this group is zero by the Lemma 5.8.2, so its quotient by the V-torsion is zero as well. □

5.9. **Surfaces with $\kappa(X) = 0$.** Let X be a smooth projective surface with $\kappa(X) = 0$. We can describe the crystalline torsion of such surfaces completely. The description of surfaces with $\kappa(X) = 0$ breaks down in to the following cases based on the value of b_2 of the surface X (see [2]).

Proposition 5.9.1. Let X/k be a smooth projective surface of Kodaira dimension zero. Then one has the following:

(1) if $b_1(X) = 4$, then X is an abelian surface and $H^2_{\text{cris}}(X/W)$ is torsion free so all species of torsion are zero; moreover X is Hodge-Witt if and only if X has p-rank one.

(2) if $b_2(X) = 22$, then X is a $K3$-surface and $H^2_{\text{cris}}(X/W)$ is torsion free and X is Hodge-Witt if and only if the formal Brauer group of X is of finite height.

(3) Assume $b_2 = 2$. Then $b_1 = 2$ and there are two subcases given by the value of p_g:
   (a) if $p_g = 0$, then $H^2_{\text{cris}}(X/W)$ has no torsion and X is Hodge-Witt;
   (b) if $p_g = 1$ then $H^2_{\text{cris}}(X/W)$ has V-torsion and $\text{Pic}(X)$ is not reduced.

(4) if $b_2 = 10$, then $p_g = 0$ and unless $\text{char}(k) = 2$ and in the latter case $p_g = 1$; in the former case X is Hodge-Witt and X has no V-torsion; if $p_g = 1$ then $H^2_{\text{cris}}(X/W)$ has V-torsion.

Proof. The assertion (1) is well-known. The assertion (2) is due to [24]. The cases when X has $p_g = 0$ can be easily dealt with by using Lemma 5.8.2 and Lemma 5.8.4. □

Corollary 5.9.2. Let X be a smooth projective surface over a perfect field. Assume $\kappa(X) = 0$ then $H^2_{\text{cris}}(X/W)$ has no exotic torsion.

Proof. The cases when $p_g = 0$ are treated by means of Lemma 5.8.4. The remaining cases follow from Suwa's criterion (see [29]) as in all these case one has by [2] that $q = -p_a$ so Suwa's criterion applies and in this situation $H^2(X, W(\mathcal{O}_X))$ is V-torsion, and therefore there is no exotic torsion □

Corollary 5.9.3. Let X be a smooth, projective surface over an algebraically closed field k of characteristic $p > 0$.

(1) If X has exotic torsion the $\kappa(X) \geq 1$.

(2) If X has V-torsion then
   (a) $\kappa(X) \geq 1$ or,
   (b) $\kappa(X) = 0$ and X has $b_2 = 2, p_g = 1$ or $p = 2$, $b_2 = 10, p_g = 1$.

5.10. **A general criterion.** Apart from [18] and [29] we do not know any useful general criteria for ruling out existence of exotic torsion. The following trivial result is often useful in dealing with exotic torsion in surfaces of general type.

Proposition 5.10.1. Let X/k be a smooth, projective surface over a perfect field. Assume $\text{Pic}(X)$ is reduced and $H^2(X, W(\mathcal{O}_X))$ is of finite type. Then $H^2_{\text{cris}}(X/W(k))$ does not contain exotic torsion.

Proof. Recall from [24] that a smooth projective surface is Hodge-Witt if and only if $H^2(X, W(\mathcal{O}_X))$ is of finite type. Then as $\text{Pic}(X)$ is reduced, we see that V is injective on $H^2(X, W(\mathcal{O}_X))$. Thus $H^2(X, W(\mathcal{O}_X))$ is a Cartier module of finite type. By [16, Proposition 2.5, page 99] we know that any R^0-module which is a finite type $W(k)$-module is a Cartier module if and only if it is a free $W(k)$-module. Thus $H^2(X, W(\mathcal{O}_X))$ is a free $W(k)$-module of finite type. By [13, Section 6.7, page 643] we see that the exotic torsion of $H^2_{\text{cris}}(X/W(k))$ is zero as it is a quotient of the image of torsion in $H^2_{\text{cris}}(X/W(k))$ (under the canonical projection $H^2_{\text{cris}}(X/W(k)) \to H^2(X, W(\mathcal{O}_X)))$ by the V-torsion of $H^2(X, W(\mathcal{O}_X))$. But as $H^2(X, W(\mathcal{O}_X))$ is torsion free, we see that the exotic torsion is zero. □

6. Mehta's question for surfaces

6.1. In this section we answer the following question of Mehta (see [18]):

Question 6.1.1. Let X/k be a smooth, projective, Frobenius split variety over a perfect field k. Then there exists a Galois étale cover $X' \to X$ such that $H^2_{\text{cris}}(X/W)$ is torsion free.

6.2. In [18] it was shown that the second crystalline cohomology of smooth, projective surface does not have exotic torsion in the second crystalline cohomology. In [19] it was shown that any smooth, projective Frobenius split surface is ordinary.

6.3. We will prove now that the answer to the above question is affirmative and in fact the assertion is true more generally for X with $\kappa(X) \leq 0$. The main theorems of this section are

Theorem 6.3.1. Let X be a smooth, projective surface of Kodaira dimension at most zero, then there exists a Galois étale cover $X' \to X$ such that $H^2_{\text{cris}}(X/W)$ is torsion free.

Theorem 6.3.2. Let X be a smooth, projective surface over a perfect field. Assume X is Frobenius split. Then there exists a Galois étale cover $X' \to X$ such that $H^2_{\text{cris}}(X'/W)$ is torsion free.

Proof. [of Theorem 6.3.1] We now note that Mehta's question is trivially true for ruled surfaces as these have torsion-free crystalline cohomology. So we may assume that $\kappa(X) = 0$. In this case we have a finite number of classes of surfaces for which the assertion has to be verified. These classes are classified by b_2. When X is a $K3$ or an Enriques' surface or an abelian surface then we can take $X' = X$ as such surfaces have torsion free crystalline cohomology. When $b_2 = 2$ the surface is bielliptic and by explicit classification of these we know that we may take the Galois cover to be the product of elliptic curves and so we are done in these cases as well. □

Proof. [of Theorem 6.3.2] After Theorem 6.3.1 it suffices to prove that the Kodaira dimension of a Frobenius split surface is at most zero. This follows from Proposition 6.3.3 below (and is, in any case, well-known to experts). □

Proposition 6.3.3. Let X be a smooth projective surface. If X is a Frobenius split then, X has Kodaira dimension at most zero and is in the following list:

(1) X is either rational or ruled over an ordinary curve,
(2) X is a either an ordinary $K3$, or an ordinary abelian surface or X is bielliptic with an ordinary elliptic curve as its Albanese variety, or X is an ordinary Enriques surface.

Proof. We first control the Kodaira dimension of a Frobenius split surface. By [22] we know that if X is Frobenius split, then $H^2(X, K_X) \to H^2(X, K_X^p)$ is injective, or by duality, $H^0(X, K_X^{1-p})$ has a non-zero section and hence in particular, $H^0(X, K_X^{-n})$ has sections for large n. Hence, if $\kappa(X) \geq 1$, then as the pluricanonical system P_n is also non-zero for large n, so we can choose an n large enough such both that K_X^n and K_X^{-n} have sections and so $K_X^n = \mathcal{O}_X$ for some integer n. But this contradicts the fact that $\kappa(X) = 1$, for in that case K_X is non-torsion, so we deduce that X has $\kappa(X) \leq 0$. Now the result follows from the classification of surfaces with $\kappa(x) \leq 0$. □

7. Hodge-Witt numbers of threefolds

In this section we compute Hodge-Witt numbers of smooth projective threefolds. In Theorem 7.3.1 we characterize Calabi-Yau threefolds with negative Hodge-Witt numbers and in Proposition 7.4.1 we provide an example of a Calabi-Yau threefold with negative Hodge-Witt numbers (the threefold in question is the Hirokado threefold).

7.1. We begin by listing all the Hodge-Witt numbers of a smooth, projective threefolds which are always non-negative.

Proposition 7.1.1. Let X/k be a smooth, projective threefold over a perfect field of characteristic $p > 0$.

1. Then $h_W^{i,j} \geq 0$ except possibly when $(i,j) \in \{(1,1),(2,1),(1,2),(2,2)\}$.
2. All the Hodge-Witt numbers except $h_W^{1,1} = h_W^{2,2}, h_W^{1,2} = h_W^{2,1}$ coincide with the corresponding slope numbers.
3. For the exceptional cases we have the following formulas.

$$h_W^{1,2} = m^{1,2} - T^{0,3} \tag{7.1.2}$$

$$h_W^{1,1} = m^{1,1} - 2T^{0,2} \tag{7.1.3}$$

Proof. Let us prove (1). This uses the criterion for degeneration of the slope spectral sequence given in [18]. The criterion shows that $T^{i,j} = 0$ unless $(i,j) \in \{(0,3),(0,2),(1,2),(3,1)\}$. By 2.24 $h_W^{i,j}$ it suffices to verify that $T^{i-1,j+1} = 0$ except possibly in the four cases listed in the proposition. This completes the proof of (1). To prove (2), we begin by observing that Hodge-Witt symmetry 2.26.1 gives $h_W^{2,1} = h_W^{1,2}$ and we also have $h_W^{1,1} = h_W^{3-1,3-1} = h_W^{2,2}$. So this proves the first part of (2). Next the criterion for degeneration of the slope spectral sequence shows that in all the cases except the listed ones, the domino numbers which appear in the definition of $h_W^{i,j}$ are zero. This proves (2). The second formula of (3) now follows again from the definition of $h_W^{i,j}$ (see 2.24 and the criterion for the degeneration of the slope spectral sequence. The first formula of (3) follows from the definition of $h_W^{1,2} = m^{1,2} + T^{1,2} - 2T^{0,3}$, and by duality for domino numbers 2.25.1 we have $T^{1,2} = T^{0,3}$. □

7.2. **Hodge-Witt numbers of Calabi-Yau threefolds.** The formulas for Hodge-Witt numbers can be made even more explicit in the case of Calabi-Yau varieties.

Proposition 7.2.1. Let X be a smooth, projective Calabi-Yau variety. Then the Hodge-Witt numbers of X are given by:

$$h_W^{0,0} = 1 \quad (7.2.2)$$

$$h_W^{0,1} = 0 \quad (7.2.3)$$

$$h_W^{0,2} = 0 \quad (7.2.4)$$

$$h_W^{0,3} = 1 \quad (7.2.5)$$

$$h_W^{1,1} = b_2 \quad (7.2.6)$$

$$h_W^{1,2} = b_2 - \frac{1}{2}c_3(X) \quad (7.2.7)$$

$$h_W^{1,3} = 0 \quad (7.2.8)$$

The remaining numbers are computed from these by using Hodge-Witt symmetry and the symmetry $h_W^{i,j} = h_W^{3-i,3-j}$.

Proof. We first note that $h_W^{0,0} = h^{0,0} = 1$ is trivial. The Hodge-Witt numbers in the first four equations are also non-negative by the previous proposition as $T^{0,2} = 0$. Moreover, by [8] it suffices to note that $h_W^{i,j} \leq h^{i,j}$ and in the second and the third formulas we have by non-negativity of $h_W^{i,j}$ that $0 \leq h_W^{0,1} \leq h^{0,1} = 0$ (by the definition of Calabi-Yau threefolds) and similarly for the third formula. The fourth formula is a consequence of Crew's formula and first three equations:

$$0 = \chi(O_X) = h_W^{0,0} - h_W^{0,1} + h_W^{0,2} - h_W^{0,3} \quad (7.2.9)$$

In particular we deduce from the fourth formula and

$$0 \leq h_W^{0,3} = m^{0,3} + T^{0,3} \leq 1$$

that if $T^{0,3} = 0$ so that X is Hodge-Witt then the definition of $m^{0,3}$ shows that

$$m^{0,3} = \sum_{\lambda} (1-\lambda) \dim H^3_{cris}(X/W)_{[\lambda]}$$

. So that the inequality shows that $H^3_{cris}(X/W)$ contains at most one slope $0 \leq \lambda < 1$ with $\lambda = \frac{h-1}{h}$ (with h allowed to be 1, to include $\lambda = 0$), and so if $T^{0,3} = 0$ then $m^{0,3} = 1$. Thus it remains to prove the formulas for $h_W^{1,1}$ and $h_W^{2,1}$. We first note that by definition:

$$h_W^{1,1} = m^{1,1} + T^{1,1} - 2T^{0,2}. \quad (7.2.10)$$

Now as $h^{0,2} = 0$ we get $T^{0,2} = 0$, and $T^{1,1} = 0$ by [16, Corollaire 3.11, page 136]. Thus we get $h_W^{1,1} = m^{1,1}$. Next

$$m^{0,2} + m^{1,1} + m^{2,0} = b_2$$

and as $m^{0,2} = 0 = m^{2,0}$ we see that $h_W^{1,1} = m^{1,1} = b_2$. The remaining formula is also a straight forward application of Crew's formula

$$\chi(\Omega^1_X) = h_W^{1,0} - h_W^{1,1} + h_W^{1,2} - h_W^{1,3} \quad (7.2.11)$$

and the Grothendieck-Hirzebruch-Riemann-Roch for Ω^1_X, which we recall in the following lemma. □

Lemma 7.2.12. Let X be a smooth projective threefold over a perfect field. Then

$$\chi(\Omega^1_X) = -\frac{3}{4}c_1^3 - \frac{23}{24}c_1 \cdot c_2 - \frac{1}{2}c_3. \quad (7.2.13)$$

Proof. This is trivial from the Grothendieck-Hirzebruch-Riemann-Roch theorem. We give a proof here for completeness. We have

$$\begin{aligned}\chi(\Omega^1_X) &= \left[3 - c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}\left(-c_1^3 - 3c_1\cdot c_2 - 3c_3\right)\right] \\ &\quad \times \left[1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1\cdot c_2\right]_3\end{aligned}$$

The subscript 3 signifies that we take the terms of degree three after multiplying the two factors out. One checks that this simplifies to the claimed equation. □

7.3. **Calabi-Yau threefolds with negative $h^{1,2}_W$.** In this section we investigate Calabi-Yau threefolds with negative Hodge-Witt numbers. From the formulas (7.2.1) it is clear that the only possible Hodge-Witt number which might be negative is $h^{1,2}_W$. We begin by characterizing such surfaces (see Theorem 7.3.1 below). Then we verify (in Proposition 7.4.1) that the in characteristic $p = 2, 3$, there do exist Calabi-Yau threefolds with negative Hodge-Witt numbers. These are the Hirokado and Schröer Calabi-Yau threefolds (which do not lift to characteristic zero).

Theorem 7.3.1. Let X be a smooth, projective Calabi-Yau threefold over a perfect field of characteristic $p > 0$. Then the following conditions are equivalent

(1) the Hodge-Witt number $h^{1,2}_W = -1$,
(2) the Hodge-Witt number $h^{1,2}_W < 0$,
(3) the W-module $H^3_{cris}(X/W)$ is torsion,
(4) the Betti number $b_3 = 0$.
(5) the threefold X is not Hodge-Witt and the slope number $m^{1,2} = 0$.

Proof. It is clear that (1) implies (2), and similarly it is clear that (3) $\Leftrightarrow$ (4). So the only assertions which need to be proved are the assertions (2) $\Rightarrow$ (3) and the assertions (4) $\Rightarrow$ (5) and the assertion (5) implies (1). So let us prove (2) implies (3). By the proof of 7.2.1 we see that $h^{1,2}_W = m^{1,2} - T^{0,3}$ and as $T^{0,3} \leq 1$, we see that if $h^{1,2}_W < 0$ then we must have $h^{1,2}_W = -1, T^{0,2} = 1, m^{1,2} = 0$ (the first of these equalities of course shows that (2) implies (1)). So the hypothesis of (2) implies in particular that $T^{0,3} = 1$ in other words, X is non-Hodge-Witt and so $H^3(X, W(\mathcal{O}_X))$ is killed by a power of p. Hence the number $m^{0,3} = 0$. Now by the symmetry (2.25.1) we see that $m^{0,3} = m^{3,0} = m^{1,2} = m^{1,2} = 0$. From this and the formula (2.25.2) we see that

$$b_3 = m^{0,3} + m^{1,2} + m^{2,1} + m^{3,0} = 0.$$

This completes the proof of (2) implies (3). Let us prove that (4) implies (5). The hypothesis of (4) and preceding equation shows that $m^{0,3} = m^{1,2} = 0$. So we have to verify that X is not Hodge-Witt. Assume that this is not the case. The vanishing $m^{0,3} = 0$ says that $H^3(X, W(\mathcal{O}_X)) \otimes_W K = 0$ and as $H^2(X, \mathcal{O}_X) = 0$ we see that V is injective on $H^3(X, W(\mathcal{O}_X))$. If X is Hodge-Witt, then so this W-module is a finite type W-module with V-injective. Therefore it is a Cartier module of finite type. By [16] such a Cartier module is a free W-module. Hence $H^3(X, W(\mathcal{O}_X))$ is free and torsion so we deduce that $H^3(X, W(\mathcal{O}_X))$ is zero. But as $H^3(X, W(\mathcal{O}_X))/VH^3(X, W(\mathcal{O}_X)) = H^3(X, \mathcal{O}_X) \neq 0$. This is a contradiction. So we see that (4) implies (5). So now let us prove that (5) implies (1). The first hypothesis of (5) implies that X is a non Hodge-Witt Calabi-Yau threefold so that

$T^{0,3} = 1$ and hence we see that $h_W^{1,2} = m^{1,2} - T^{0,3} = -1 < 0$. This completes the proof of the theorem. □

7.4. **Remark.** Such a characterization does not seem possible for Calabi-Yau varieties of dimensions greater than three.

Proposition 7.4.1. Let k be an algebraically closed field of characteristic $p = 2, 3$. Then there exists smooth, projective Calabi-Yau threefold X such that $h_W^{1,2} < 0$.

Proof. In [12], [26] one finds examples of (and in fact, in the latter case, families of) Calabi-Yau threefolds in characteristic $p = 2, 3$ which are not liftable to characteristic zero. We claim that these Calabi-Yau threefolds are the examples we seek. It was verified in loc. cit. that these threefolds have $b_3 = 0$. So we are done by Theorem 7.3.1. □

Corollary 7.4.2. The Hirokado and Schröer threefolds are not Hodge-Witt.

Remark 7.4.3. The Hirokado and Schröer threefolds have been investigated in detail by Ekedahl (see [9]) who has prove their arithmetical rigidity. One should note that the right hand side of (7.2.1) is non-negative if X lifts to characteristic zero without any additional assumptions on torsion of $H^*_{cris}(X/W)$ as the following proposition shows.

Proposition 7.4.4. Let X be a smooth, projective Calabi-Yau threefold. If X lifts to characteristic zero then $b_2 \geq \frac{1}{2}c_3$.

Proof. Under the hypothesis, we know that $b_1 = b_5 = 0$ so that $c_3 = 2 + 2b_2 - b_3$, so that $b_2 - \frac{1}{2}c_3 = \frac{b_3}{2} - 1$ and by the Hodge decomposition, $b_3 \geq 2$ is even and so the assertion holds. □

Conjecture 7.4.5. Let X be a smooth projective Calabi-Yau threefold. Then

(1) X lifts to characteristic zero if and only if $h_W^{1,2} \geq 0$.
(2) $H^2_{\mathrm{cris}}(X/W)$ is torsion free.

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