SOME REMARKS ON VECTOR BUNDLES

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1. Introduction

Let X be a smooth projective variety over an algebraically closed field k of characteristic $p > 0$, $F : X \to X$ the absolute Frobenius morphism, fix an ample line bundle H on X. If V is a vector bundle, then we will say that V is Frobenius stable (resp. frobenius semi-stable) with respect to H if V and $F^*(V)$ are stable (respectively. semi-stable) with respect to H.

In general, if V is stable (or semi-stable), $F^*(V)$ need not be stable or semi-stable–in other words, Frobenius ascent disturbs stability. On the other hand, if V is stable and if V descends under F, i.e., if there exists a vector bundle V_1 on X such that $V = F^*(V_1)$, then V_1 is stable. These seemingly opposite aspects of the Frobenius morphism are exploited in this note. In Section 2, Section 3 we provide refinements (which may be of independent interest) of known results, the techniques of these two sections are used without further mention in Section 4 which contains the main result of this paper (see Theorem 4.2). The main question of interest to us is the following: assume X is a smooth projective variety and suppose V is a vector bundle which descends under Frobenius modulo an infinite set of primes then is it true that E is semistable (with respect to any ample line bundle on X)? In Theorem 4.2 we prove this for rank two vector bundles on X under the assumption that $\mathrm{Pic}(X) = \mathbf{Z}$ and $\mathrm{Pic}^0(X) = 0$ (scheme theoretically). In fact we prove more: we show that under these assumptions any rank two vector bundle is either trivial or stable. Note that unlike [5] we do not assume the existence of a global integrable connection on our vector bundle E. Needless to say that this is inspired by Grothendieck's conjecture and [5].

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2. Frobenius ascent

2.1. **Preliminaries.** In this section, let X be a smooth projective variety over an algebraically closed field k of characteristic $p > 0$. We make the following hypothesis (not necessarily independent):

Hypothesis 1 (D-I). *Assume that* $\mathrm{Pic}(X) \simeq \mathbf{Z}$*; X has a flat lifting to $W_2(k)$. Further assume that $p > \dim(X)$.*

Hypothesis 2 (KAN). *Assume that* $\mathrm{Pic}(X) = \mathbf{Z}$ *and X satisfies Kodaira-Akizuki-Nakano vanishing.*

Observe that hypersurfaces in projective space are examples of varieties satisfying these hypothesis. And indeed by [2], any variety satisfying Hypothesis D-I, also satisfies Hypothesis KAN.

In addition we need to recall from [6], the notion of a Harder-Narasimhan filtration for rank two vector bundles. Suppose V is a non-semi-stable rank two bundle on X. Then there exists a torsion free sub sheaf (of rank one) $L \hookrightarrow V$ with the following properties: (1) V/L is also torsion free (of rank one), (2) $\deg(L) = \mu(L) > \mu(V/L) = \deg(V/L)$. Moreover, such a filtration on V is unique.

2.2. Our basic technique is illustrated in the following mild refinement of the results of [7].

Theorem 2.1. *Let X be a smooth projective variety satisfying either Hypothesis D-I or Hypothesis KAN, and let V be any rank two semi-stable bundle on X. Then $F^*(V)$ is semi-stable.*

Proof. Assume, if possible, that $F^*(V)$ is not semi-stable. Let

$$0 \to L \to F^*(V) \to F^*(V)/L \to 0 \tag{2.1}$$

be the Harder-Narasimhan filtration of $F^*(V)$. Let $\mu_1 = \mu(L)$ and $\mu_2 = \mu(F^*(V)/L)$. then by the properties of the Harder-Narasimhan filtration we see that $\mu_1 > \mu_2$.

We try and descend this filtration under Frobenius. As is pointed out in [4], $F^*(V)$ carries a connection, whose flats sections are isomorphic to V. Descent under Frobenius of a subsheaf thus amounts to studying the behavior of the sub sheaf under the natural connection. Then it is well known that the obstruction to descending this subsheaf under this connection on $F^*(V)$ is given by the second fundamental form [7]

$$T_X \to \mathrm{Hom}(L, F^*(V)/L). \tag{2.2}$$

By Lemma 1.1.16 of [1] we see, as $F^*(V)/L$ is torsion free, that L is normal and hence by Lemma 1.1.12 (loc. cit), it is reflexive. Then by Lemma 1.1.15 (loc. cit) we see that L is a line bundle.

Moreover on passing to double duals we see that second fundamental form gives a map from $T_X \to G$ where G is locally free of rank one and has degree equal to $-\mu_1 + \mu_2 < 0$. Dualizing, we get a map $G^{-1} \to \Omega^1_X$, or equivalently a section of $\Omega^1_X \otimes G$. But G has negative degree (and hence by hypothesis) is negatively ample. As X satisfies Hypothesis D-I or Hypothesis KAN, we see that X satisfies Kodaira vanishing theorem; so we see that this section is zero. □

Corollary 2.2. *In addition to the conditions of the above theorem, assume that $H^0(X, \Omega^1_X) = 0$ then every stable rank two bundle V on X is also Frobenius-stable.*

Proof. By the above theorem $F^*(V)$ is certainly semi-stable. So assume if possible that there is a sub-sheaf (torsion-free) L of rank one such that $\mu(L) = \mu(F^*(V))$. Then we get a map

$$T_X \to \mathrm{Hom}(L, F^*(V)/L) \tag{2.3}$$

and one checks that $\deg(L) = \deg(F^*(V)/L) = \deg(F^*(V))/2$. Hence we get, on dualizing, a section of Ω^1_X. But as $H^0(X, \Omega^1_X) = 0$ we see that this section must be zero and so L descends under Frobenius to give a sub sheaf of V and slope equal to V, but as V is stable this is not possible. This completes the proof. □

3. Stratified Bundles

3.1. Let V be a vector bundle on X. We say, following [3] that V is *stratified bundle* or a *flat bundle* if, there exists a sequence of vector bundles $\{V_n\}_{n\geq 0}$ on X such that $V = V_0$ and $F^*(V_i) = V_{i-1}$. By a theorem of N. Katz (see Theorem 1.3 of [3]), this data is equivalent to saying that V is a module for the ring of differential operators, $\mathcal{D}_X = \mathrm{Diff}(\mathcal{O}_X, \mathcal{O}_X)$, on X.

The following proposition is certainly well-known to experts but we include it here for completeness as we do not know an explicit reference.

Proposition 3.1. *Let X be a smooth projective variety over a perfect field of characteristic $p > 0$. Suppose that V is a stratified bundle on X. Fix an ample line bundle H on X. Then there exists some descent V_i of V which is semi-stable with respect to H.*

Proof. If V is semistable with respect to H then there is nothing to prove. So assume that V is not semistable with respect to H. Then we have an associated Harder-Narasimhan filtration of V. The main point is that the degrees of all the graded pieces of the filtration (with respect to H) are fixed. Now assume that no V_i is semistable with respect to H. Then we get an infinite collection of Harder-Narasimhan filtrations one for each V_i and pulling them back by appropriate powers of Frobenius we see that V has destablising subbundles of unbounded degrees. This is a contradiction as the Harder Narasimhan filtration of V is uniquely defined by its properties (and the degree are bounded). This proves the assertion. □

In [3], Gieseker had asked if on a simply connected, projective variety any stratified bundle is trivial. A small modification of the technique used in the proof above also gives leads to following two theorems in the direction of Gieseker's question.

3.2. We are now ready to prove:

Theorem 3.2. *Suppose that X satisfies either Hypothesis D-I or Hypothesis KAN and $\mathrm{Pic}^0(X) = 0$ as a scheme. Let V be a stratified bundle of rank two on X. Then the following dichotomy holds:*

(1) *either V is trivial or,*
(2) *stable.*

and all the Frobenius pull-backs of V are semi-stable.

Proof. We first show that V and all its Frobenius pull-backs are semi-stable. We begin by noting that as V is stratified, the chern classes of V are divisible by p^n for all n by Lemma 2.1 of [3]. This divisibility shows that the chern classes are numerically trivial and this together with the assumption that $\mathrm{Pic}(X) = \mathbf{Z} = \mathbf{Z}[\mathcal{O}_X(1)]$ implies that $c_1(V) = 0$ in particular we have $\det(V) = \mathcal{O}_X$, and $c_2(V)$ is numerically trivial and in particular $c_2(V) \cdot \mathcal{O}_X(1)^{\dim(X)-2} = 0$ for an ample generator $\mathcal{O}_X(1)$ of $\mathrm{Pic}(X)$.

As V is stratified, we see that $V = F^*(V_1)$ for some V_1 on X. Thus in particular V carries a connection. So we apply the descent argument developed in the previous section to descend the Harder-Narasimhan filtration of V under Frobenius. This is again possible, because of Kodaira vanishing theorem (for X). Thus the Harder-Narasimhan filtration on V descends to V_1. In particular, we see that the degree of the first step L of the Harder-Narasimhan filtration is divisible by p. Repeating this argument with $V_1, V_2, \ldots$ we see that the degree of L is divisible by larger and larger powers of p. But the degree of the destabilizing sub sheaf L on V is positive and bounded above (by the uniqueness of the Harder-Narasimhan filtration), and hence cannot be divisible by arbitrary powers of p. Thus we have a contradiction. Hence V must be semi-stable. As V is stratified, we see that $F^*(V)$ is also stratified and so $F^*(V)$ is also semi-stable. Repeating this argument we see that all Frobenius pull-backs of V are semi-stable.

Now the remaining part of the theorem is proved as follows. If V is trivial or stable there is nothing to do. So assume that V is non-trivial and non-stable.

Then we have an exact sequence

$$0 \to \mathcal{O}_X \to V \to Q \to 0. \tag{3.1}$$

This follows from the following facts: the first step of the Jordan-Holder filtration on V is reflexive of rank one and hence locally free, further from degree considerations, this line bundle has degree zero; and lastly by hypothesis we have $\operatorname{Pic}(X) = \mathbf{Z}$ and hence any degree zero line bundle is trivial.

If Q is locally free then $Q = \mathcal{O}_X$ and so we have an extension

$$0 \to \mathcal{O}_X \to V \to \mathcal{O}_X \to 0 \tag{3.2}$$

and as $\operatorname{Pic}(X) = \mathbf{Z}$ and the scheme $\operatorname{Pic}^0(X) = 0$ we see that $H^1(X, \mathcal{O}_X) = 0$ so any such extension is trivial.

So assume that Q is not locally free. As Q is torsion free Q injects into its double dual $Q \hookrightarrow Q^{**}$ (see [1, page 148]) and note in fact that the dual Q^* of Q is reflexive of rank one (see [1, page 150]). Hence Q^* and Q^{**} are locally free of rank one. And so writing $\mathcal{I} = Q \otimes Q^{**-1}$ we see that $\mathcal{I}$ is an ideal sheaf of a subscheme of X. Further it is clear from a chern class computation that $\mathcal{I}$ does not contribute to $c_1(V) = \mathcal{O}_X$ thus we have $c_1(Q^{**}) = 0$ and so $Q^{**} = \mathcal{O}_X$. Thus $Q = \mathcal{I}$ is the ideal sheaf of a closed subscheme of X.

Now we claim that $\mathcal{I} = \mathcal{O}_X$. Suppose this is not true. Computing $c_2(V) = c_2(Q) = c_2(\mathcal{I})$ and we know that $c_2(V)$ is numericaly trivial. Thus we have a closed subscheme of X which is numerically trivial. Now using formulas in [9, page 106, 108] once checks that this is not possible. So $Q = \mathcal{I} = \mathcal{O}_X$ and so as $H^1(X, \mathcal{O}_X) = 0$ we see that V splits but this contradicts our hypothesis that V is not trivial. □

3.3. Under a different hypothesis X, i.e. those of [7], we can prove the following.

Proposition 3.3. *Assume that the tangent bundle of X is stable of positive degree and* $\operatorname{Pic}(X) = \mathbf{Z}$. *Then every stratified bundle V (need not be rank two) on X is Frobenius-semi-stable.*

Proof. By [7], Theorem 2.1, any stable (resp. semi-stable) vector bundle on X is Frobenius-stable (resp. Frobenius-semi-stable). Moreover, under the hypothesis on X, from [7] and [10], tensor products of semi-stable vector bundles on X is again Frobenius semi-stable.

As was remarked before, V has degree zero. Assume if possible that V is not semi-stable. Then V has a Harder-Narasimhan filtration. Let $W \hookrightarrow V$ be a destabilizing sub sheaf of V, with W of positive degree. Then we get a map $T_X \to \operatorname{Hom}(W, V/W)$. Moreover, the target carries a Harder-Narasimhan filtration and the graded pieces are semi-stable of negative degrees. As the T_X is stable we see that any such map is zero. Thus we see that W descends under the Frobenius, and by repeating this argument we see that degree of W is divisible by arbitrarily large powers of p and hence W has degree zero, which is a contradiction. Thus V is semi-stable. As $F^*(V)$ is stratified it is also semi-stable. □

Remark 3.4. *Suppose k is a finite field. Assume that X satisfies either Hypothesis D-I or Hypothesis KAN. Let us suppose that vector bundles with fixed chern classes on X move in a bounded family. Then for a stratified bundle V, we get an infinite set of points on the moduli space of semi-stable vector bundles with trivial chern classes on X, corresponding to the semi-stable bundles $V, F^*(V), (F^2)^*(V), \ldots$. But over a finite field, the moduli consists of finite set of points. Thus for some m, n we must have $(F^m)^*(V) = (F^n)^*(V)$ and $m > n$. Now by a well-known Theorem of Mumford, $(F^n)^*(V)$ must be a finite vector bundle, i.e, it becomes trivial on a finite étale cover of X. If X is simply-connected we see that $(F^n)^*(V)$ must be trivial.*

There are situations where boundedness is known. In particular, by [6] *for any smooth projective variety X and rank two bundles on X, we know boundedness of families of stable vector bundles and so in this case every stratified bundle on X pulls back under a suitable power of Frobenius map to a trivial bundle.*

4. Frobenius Descent

4.1. **Notations.** For this section we assume that our ground field is $\mathbf{C}$ the field of complex numbers. We assume as before, that X is a smooth projective variety over $\mathbf{C}$, with $\mathrm{Pic}(X) = \mathbf{Z}$. Let V be a vector bundle on X. We may assume that X, V descend to a finitely generated field $K \subset \mathbf{C}$. Further we may assume that X gives rise to a scheme $\mathcal{X} \to \mathrm{Spec}(R)$ where R is a finite type $\mathbf{Z}$ sub-algebra of K and that the generic fibre of $\mathcal{X}$ is isomorphic (over K) to X, by shrinking R if necessary, we may assume that $\mathcal{X} \to \mathrm{Spec}(R)$ is smooth and projective, and V gives a vector bundle $\mathcal{V}$ on $\mathcal{X}$, which is isomorphic to V at the generic point of R. We will say that V descends under Frobenius at a geometric point of R if the restriction of $\mathcal{V}$ to the fibre over this point descends under the Frobenius. We note that as R is of finite type over $\mathbf{Z}$, all closed points are necessarily live in positive characteristic. In what follows we may need to replace R by Zariski open subsets (non-empty!). The main question of interest to us is the following (we may think of it as a variant of Grothendieck's p-curvature conjecture–see [5]).

Question 4.1. *In the notation and terminology set up above, let X be a smooth, projective variety over complex numbers and assume that V is a vector bundle such that for an infinite number of primes p, the reduction modulo p of V descends under Frobenius, then is V semistable with respect to any ample line bundle of X?*

4.2. The following theorem is the main theorem of this paper. In it we answer Question 4.1 for rank two bundles under additional assumptions on X.

Theorem 4.2. *In the notations of the previous paragraph. Let X be a smooth projective variety with* $\mathrm{Pic}(X) = \mathbf{Z}$ *(as a scheme) and, let V vector bundle of rank two on X. Assume further that V descends under Frobenius for an infinite number of geometric points of R of distinct residue characteristics. Then V is either trivial or stable. Moreover if X is simply connected then V is trivial.*

Proof. Fix an ample generator $\mathcal{O}_X(1)$ for $\mathrm{Pic}(X) = \mathbf{Z}$. We show that if V is non-trivial then it must be stable and with $c_1(V) = 0$ and $c_2(V)$ is numerically trivial and in particular $c_2(V) \cdot \mathcal{O}_X(1)^{\dim(X)-2} = 0$. These two properties together stability of V imply, by the analogue of the Narasimhan-Seshadri theorem in higher dimensions due to V. B. Mehta and A. Ramanathan (see [8]), that the bundle V comes from a representation of the fundamental group and hence will be trivial if the fundamental group of X is trivial.

We proceed as in the previous section. As V descends under Frobenius for an infinity of closed points, we see that $c_1(V) = 0$. This is immediate from the divisibility of chern classes implied by the descent properties of V. This also gives us $c_2(V)$ is numerically trivial and in particular $c_2(V) \cdot \mathcal{O}_X(1)^{\dim(X)-2} = 0$.

Assume if possible that V is not semi-stable, let L be the destabilizing sub sheaf of maximal degree. We recall that stability is an open condition, so by shrinking R if necessary, we may assume that this choice of the Harder-Narasimhan filtration spreads out to R in such a way that it restricts to give a Harder-Narasimhan filtration on each fibre of the projection $\mathcal{X} \to \mathrm{Spec}(R)$. It is important to note that the Harder-Narasimhan filtration is defined by global (i.e. over R) sheaves. As V has numerically trivial chern classes, we see that V has degree zero and in particular that L has positive degree on the generic fibre and hence is ample; the double dual of V/L is then a line bundle of negative degree and hence is anti-ample. We may shrink R further to assume that $L, V/L$ have these properties on geometric fibres.

By the argument as above and by Kodaira-Akizuki-Nakano vanishing (which is valid for almost all closed points of R), we see that the sub sheaf L descends under Frobenius for an infinity of geometric points of R. Hence degree of L is divisible by an infinity of primes and hence is unbounded. But L has bounded degree. Hence we see that degree of L must be zero, which is a contradiction. Thus V is semi-stable with numerically trivial chern classes.

Now suppose that V is non-trivial and non-stable. Then we have an exact sequence

$$0 \to \mathcal{O}_X \to V \to Q \to 0 \tag{4.1}$$

as before. We argue as before: we observe that if Q is locally free with $Q = \mathcal{O}_X$ and so V is trivial. If Q is not locally free, then noting that $Q^{**} = \mathcal{O}_X$ we get that Q is an ideal sheaf of a closed subscheme of X. Now using formulas in [9, page 106, 108] once checks that this contradicts the numerical triviality of $c_2(V)$. This implies that $Q = \mathcal{O}_X$. So we see that $Q = \mathcal{O}_X$ and the extension splits which contradicts our assumption.

As V is stable with $c_1(V) = 0$ and $c_2(V) \cdot \mathcal{O}_X(1)^{\dim(X)-2} = 0$ then by a theorem of Mehta-Ramanathan (see [8]) V comes from a representation of the fundamental group of X and hence when X is simply connected V must be trivial. □

Remark 4.3. *As was pointed out in* [5]*, the naive analogue of Grothendieck's conjecture leads to delicate arithemetical questions when* $\mathrm{Pic}^0(X) \neq 0$*–see for instance Katz's analysis of the conjecture for elliptic curves (see Section 7 of* [5]*). From this point of view this hypothesis* $\mathrm{Pic}^0(X) = 0$ *is natural if we want to prove the bundle is stable or trivial.*

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