Cloud modeling and Lotka-Voltera

Spence Lunderman
December 9, 2017
University of Arizona
Department of Mathematics

1 Low dimensional cloud modeling

Cloud modeling is a very difficult task: one must understand the cloud's microphysics, albedo or ability to reflect sunlight, mixture of liquid and ice, etc. These difficulties result in wide variations of how clouds are represented in global climate models [2]. For example, certain cloud models that covert water to ice at relatively low temperatures lead to a global climate model with a higher mean-state cloud fraction [3].

In an attempt to tackle some of these known difficulties, researchers are looking to qualitatively describe clouds with low dimensional models. An example of this comes from Koren and Fiengold (2011): they use the Lotka-Volterra predator prey equations to model the interaction between the rain rate of a cloud and the cloud's liquid water path (column integrated liquid water content). It would seem that these two observables have some predator prey interactions. As water droplets form, they consume the water vapor within the cloud, potentially leading to the destruction of the cloud. In the absence of rain, the amount of water vapor can grow until reaching a critical point.

Figure 1 shows a cloud system over the Atlantic ocean. In the figure, there are areas of dense clouds structures and areas of open cells which show the dark ocean below. A goal of this work is to capture qualitative details of these cloud types using these predator prey insights.

As a cloud presents rain, the liquid water path (LWP) and height of the cloud (H) decrease. However, instead of setting up the predator-prey model for the interactions between LWP and R, the authors opt to design the system around H and the cloud drop concentration N. H relates to LWP by

$$LWP = \frac{c_1}{2} H^2 \tag{1}$$

where c_1 is a function of cloud based temperature and pressure and to first order the rain rate R can be expressed as

$$R = \alpha \frac{H^3}{N} \tag{2}$$

for appropriate α. The balance equations for cloud depth H and cloud drop concentration N are

$$\frac{dH}{dt} = \frac{H_o - H(t)}{\tau 1} - \dot{H}(t-T) \text{ and } \frac{dN}{dt} = \frac{N_o - N(t)}{\tau 2} - \dot{N}(t-T). \tag{3}$$

In the above equations H_o and N_o are the full environmental potential for cloud development; this means the first term in both equations represent the system's approach to their full potential with characteristic time constants $\tau 1$ and $\tau 2$. $\dot{H}$ and $\dot{N}$ are the loss of H and N after some delayed time T. Based on theoretical and empirical results, the authors derive $\dot{H}$ and $\dot{N}$ to yield the final set of equations

$$\frac{dH}{dt} = \frac{H_o - H(t)}{\tau 1} - \frac{\alpha H^2(t-T)}{c_1 N(t-T)} \quad \text{and} \quad \frac{dN}{dt} = \frac{N_o - N(t)}{\tau 2} - \alpha c_2 H^3(t-T). \tag{4}$$

For the specified parameters, figure 1 illustrates the interactions between the cloud depth and drop concentration as well as the derived interactions between LWP and rainrate.

Our goal now is to find the system's steady states and understand their stabilities. To make life simpler, we perform the following transformations to yield dimensionless solutions which we can later transform back: Let

$$t \mapsto \frac{t}{\tau_1}, \quad H \mapsto 0.01 * c_2 H, \quad \text{and} \quad N \mapsto \frac{c_2^2}{1\text{E}11 \alpha \tau_1} N \tag{5}$$

and define $c = \frac{c_2}{1\mathbf{E}9 c_1}$, $\tau = \frac{\tau_1}{\tau_2}$. This leaves us with the following delay differential equations

$$\frac{dH}{dt} = H_o - H(t) - \frac{cH^2(t-T)}{N(t-T)} \quad \text{and} \quad \frac{dN}{dt} = \tau(N_o - N(t)) - H^3(t-T). \tag{6}$$

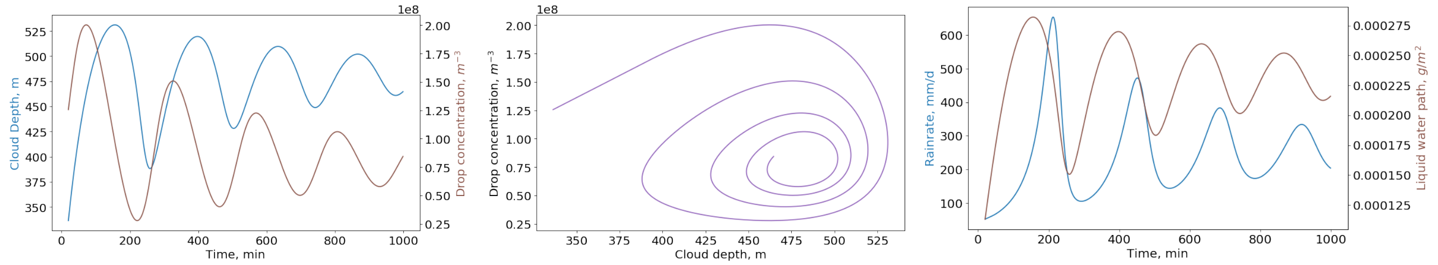

Figure 1: **Left:** Cloud depth H (blue) and cloud drop concentration N (brown). **Middle:** Cloud depth plotted against cloud drop concentration. **Right:** Rainrate (blue) and liquid water path (brown); these values are calculated by (2) and (1) respectively. For all three figures, the parameters are $H_o = 650m$, $N_o = 469\mathbf{E}6\, m^{-3}$, $\alpha = 2\mathbf{E} - 3\, m^{-5}/d$, $c_1 = 2\mathbf{E} - 9\, m^{-1}$, $c_2 = 0.3\, m^{-1}$, $\tau_1 = 80\, min$, and $\tau_2 = 84\, min$, $T = 20\, min$.

A steady state solution to the above delay differential equation is independent of the delay T. We solve for the steady state solutions in terms of H_o and N_o keeping in mind our larger parameter estimation goals. Solving the equations $\frac{dH}{dt} = 0$ and $\frac{dN}{dt} = 0$ and only keeping positive solutions we get

$$H = \frac{-N + \sqrt{N^2 + 4H_o N}}{2} \quad \text{and} \quad N = N_o - \frac{H^3}{\tau}. \tag{7}$$

Given that we are treating H_o and N_o as free parameters, the contour plots for the steady state solutions are a surface in the space $(H_o; N)$ for the cloud depth H and $(N_o; H)$ for the drop concentration N. The contour plots for the steady state solutions can be seen in figure 2.

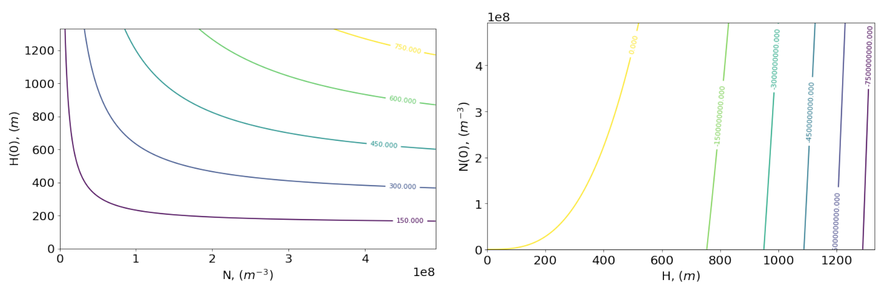

Figure 2: Steady state contour plots for: cloud depth, m, (left) and drop concentration, m^{-3}, N (right). The parameters and initial conditions are the same as figure 1.

For the cloud depth contour plot in figure 2, there appear to be two regimes: for higher drop concentration the steady states are mainly dictated by H_o and lower drop concentration seems to control the steady solution with little influence from H_o. As for the drop concentration contour plot, the steady state solutions are seemly dictated by the cloud depth, influenced little by N_o.

Studying stability of steady states and doing bifurcation analysis for delay differential equations is a difficult problem. For example, determining linear stability for a differential equations requires finding the eigenvalues of the linearized system. For delay differential equations, the characteristic equation for the linearized system can have infinitely many roots, often making it a difficult task to determine stability. For these reasons, we step away from the derived cloud models and focus our attention towards a slightly easier problem. We will now study the steady states and their stability for the delayed Lotka-Voltera equations. It is our hopes that this work can direct future work with these cloud models.

2 Predator Prey

The Lotka-Voltera equations are a standard mathematical model to illustrate the interactions between a predator and its prey. Independently derived by Alfred J. Lotka and Vito Volterra over a century ago, these equations and many variations have been studied extensively in biological and ecological sciences. The equations are given by

$$\frac{dC}{dt} = C(\alpha - \beta R) \quad \text{and} \quad \frac{dR}{dt} = -R(\gamma - \delta C)$$

where C and R represent the predator and prey population respectively and $\alpha, \beta, \gamma,$ and δ are positive model parameters. Complexity can be adding to these models by taking into account the amount of time it between birth and maturity of the predator or prey. Adding this into the model yields the equations

$$\frac{dC}{dt} = C[\alpha - \beta R(t-T)] \text{ and } \frac{dR}{dt} = -R[\gamma - \delta C(t-T)] \tag{8}$$

where T represent some time delay for the predator and prey. These are the equations that we will work with to determine the steady states and their linear stability, in an effort to guide future work with the cloud models.

This system has two biologically relevant steady states: one at $(C, R) = (0, 0)$ and the nontrivial one at $(C, R) = (\gamma/\delta, \alpha/\beta)$ which can easily derived. For the trivial steady state, we can easily determine the linear stability.

$$\begin{pmatrix} \dot{C} \\ \dot{R} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} C \\ R \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C(t-T) \\ R(t-T) \end{pmatrix} \tag{9}$$

The characteristic matrix for this linearizion about $(0, 0)$ is

$$\begin{aligned} \Delta(\lambda) &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} e^{-T\lambda} \\ &= \begin{pmatrix} \lambda - \alpha & 0 \\ 0 & \lambda + \gamma \end{pmatrix}. \end{aligned}$$

Then the characteristic equation is given by

$$P(\lambda) = \det\left(\Delta(\lambda)\right) = (\lambda - \alpha)(\lambda + \gamma) \tag{10}$$

and the roots to (11) are the desired eigenvalues: α and $-\gamma$. Given that all model parameters are positive, we know that we have a saddle point at the fixed point $(0, 0)$. For the second relevant steady state $(\gamma/\delta, \alpha/\beta)$, the linearized of (8) about the steady state is

$$\begin{pmatrix} \dot{C} \\ \dot{R} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C \\ R \end{pmatrix} + \begin{pmatrix} 0 & \frac{-\beta\gamma}{\delta} \\ \frac{\delta\alpha}{\beta} & 0 \end{pmatrix} \begin{pmatrix} C(t-T) \\ R(t-T) \end{pmatrix} \tag{11}$$

yielding the characteristic matirx

$$\Delta(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \frac{-\beta\gamma}{\delta} \\ \frac{\delta\alpha}{\beta} & 0 \end{pmatrix} e^{-T\lambda} \tag{12}$$

$$= \begin{pmatrix} \lambda & \frac{\beta\gamma}{\delta} e^{-T\lambda} \\ -\frac{\delta\alpha}{\beta} e^{-T\lambda} & \lambda \end{pmatrix}. \tag{13}$$

and characteristic equation

$$P(\lambda) = \lambda^2 + \alpha\gamma e^{-2T\lambda}. \tag{14}$$

There are infinitely many roots to (14). To begin understand the stability of the steady state, we first consider two special cases. The simplest case to consider is if $\lambda = 0$. In this case

$$P(0) = \alpha\gamma.$$

Given that α and γ are positive, $P(0)$ does not have any roots and we no longer need to consider this case. A more interesting case is if the eigenvalue is purely imaginary, i.e. $\lambda = i\omega$ for $\omega > 0$. Then (14) becomes trying to solve

$$(i\omega)^2 + \alpha\gamma e^{-2T\omega i} = 0. \tag{15}$$

Expanding (15) and splitting into real and imaginary parts, we get

$$\begin{aligned} 0 =& \alpha\gamma\cos\big(-2\omega T\big) - \omega^2 && (16)\\ \text{and } \ 0 =& \alpha\gamma\sin\big(-2\omega T\big). && (17)\end{aligned}$$

Given that α and γ are positive, (17) gives that $\omega = \dfrac{\pi}{2T}k$ for $k \in \mathbb{Z}$. Substituting $\omega = \pi/2T$ into (16),

$$\begin{aligned} & 0 = \alpha\gamma\cos\big(-2\pi\big) - \Big(\frac{\pi}{2T}\Big)^2 && (18)\\ \Rightarrow \quad & T = \frac{\pi}{2\sqrt{\alpha\gamma}} && (19)\\ \Rightarrow \quad & w = 2\sqrt{\alpha\gamma} && (20)\end{aligned}$$

Thus, $\lambda = (2\sqrt{\alpha\gamma}\,k)i$ for $k \in \mathbb{Z}\backslash\{0\}$ is a root of (14) provided the delay $T = \dfrac{\pi}{2\sqrt{\alpha\gamma}}$

Let us define a function $\lambda(T) := \mu(T) + i\omega(T)$ around $T_* = \dfrac{\pi}{2\sqrt{\alpha\gamma}}$ such that $\mu(T_*) = 0$ and $\omega(T_*) = 2\sqrt{\alpha\gamma}$. With hopes to satisfy the Hopf bifurcation theorem conditions [4] [5], we want to show that

$$\left.\frac{d\mathrm{Re}(\lambda)}{dt}\right|_{\lambda=\lambda(T_*)} > 0. \tag{21}$$

If we can show (21), it would be one step towards showing that (8) has a Hopf bifurcation at $(C, R) = (\gamma/\delta, \alpha/\beta)$. We implicitly differentiate (10) equated to zero

$$\begin{aligned} & 0 = 2\lambda\frac{d\lambda}{dt} - 2\alpha\gamma T e^{-2T\lambda}\frac{d\lambda}{dt} && (22)\\ \Rightarrow \quad & 0 = \frac{d\lambda}{dt}(2\lambda - 2\alpha\gamma T e^{-2T\lambda}) && (23)\end{aligned}$$

Evaluating (23) at $\lambda = \lambda(T_*)$ and $T = T_*$ yields

$$0 = \frac{d\lambda}{dt}(4\sqrt{\alpha\gamma}i - \pi\sqrt{\alpha\gamma}). \tag{24}$$

The right factor is nonzero implying that $\dfrac{d\lambda}{dt} = 0$ resulting in

$$\left.\frac{d\mathrm{Re}(\lambda)}{dt}\right|_{\lambda=\lambda(T_*)} = 0.$$

Thus, we can not use the Hopf bifurcation theorem to gain understanding of our dynamical system.

We focus our attention towards understanding, in the most general case, the linear stability of the steady state solution $(C, R) = (\gamma/\delta, \alpha/\beta)$. We need to understand the roots of (14). Taking small bites, we assume that $\alpha\gamma = 1$ and $T = 1/2$ leaving us to find the solutions to

$$0 = \lambda^2 + e^{-\lambda} \tag{25}$$

With an eye towards determining stability, we focus on the sign of the real parts of solutions to (25) which are of the form

$$\mathrm{Re}(\lambda) = \mathrm{Re}\left[W\left(k, \frac{-i}{2}\right)\right] \tag{26}$$

Where $W\left(k, \frac{-i}{2}\right)$ is the k^{th} solution of the Lambert W function which is defined as the inverse function to

$$f(z) = ze^z.$$

A few evaluations gives us all the information we need:

$$\mathrm{Re}\left[W\left(0, \frac{-i}{2}\right)\right] \approx 0.1626 \quad \text{and} \quad \mathrm{Re}\left[W\left(1, \frac{-i}{2}\right)\right] \approx -1.8267$$

which shows that this steady state is linearly unstable.

3 Conclusions

To gain insight into cloud modeling, a notoriously difficult task, we considered a simplified model that captures qualitative cloud behavior. The Lotka-Volterra predator prey models provide a well studied system to derive more complex dynamical systems that present oscillatory behavior. Although this work is not complete, studying the stability of steady states in the delayed Lotka-Volterra models provide useful insight. This instability of the fixed point is not physically realistic in that a predator prey system cannot have unbounded oscillations. Hopefully future work will find stable steady states in the Koren-Fiengold cloud models that match observable cloud patterns.

References

[1] Koren I., Feingold G., (2011) Aerosol–cloud–precipitation system as a predator-prey problem. Proceedings of the National Academy of Sciences.

[2] Yan, W. (2016), Modeling the effects of clouds on climate, Eos, 97, doi:10.1029/2016EO049999

[3] McCoy, Daniel T. and Tan, Ivy and Hartmann, Dennis L. and Zelinka, Mark D. and Storelvmo, Trude (2016) On the relationships among cloud cover, mixed-phase partitioning, and planetary albedo in GCMs. Journal of Advances in Modeling Earth Systems

[4] Hale, Jack K., and Huseyin Kocak. Dynamics and Bifurcations. New York, NY: Springer New York, 1991. Print.

[5] Guckenheimer, J. and Holmes, P. (2002). Nonlinear oscillations, dynamical systems, and bifurcations of vector fields. New York: Springer.

[6] Koen E., Tatyana L., Dirk R. (2000) Numerical bifurcation analysis of delay differential equations. Journal of Computational and Applied Mathematics.

[7] Forde J. (2005) Delay Differential Equation Models in Mathematical Biology. [] Kalmar-Nagy T. (2009) Stability analysis of delay-differential equations by the method of steps and inverse Laplace transform. Differential Equations and Dynamical Systems