MATH 557A: Extended Abstract on Discrete Painlevé II

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13th December, 2017

Abstract for the (Extended) Abstract

The discrete Painlevé equations are considered the fundamental objects of study in discrete integrable systems. In this extended abstract, I'll work with DP_{II} *(Discrete Painlevé II), and demonstrate some techniques which are useful in studying these sorts of discrete dynamical system. We'll see how one can compactify dynamics on* $\mathbb{R}^2$ *via projectivisation, and how one can then resolve singularities via the algebro-geometric technique of blowing up.*

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1 $\mathbf{DP_{II}}$: As an Iterated Map on $\mathbb{R}^2$

The discrete Painlevé II equation is the following recurrence relation

$$u_{n+1} + u_{n-1} = \frac{z_n u_n + a}{1 - u_n^2} \tag{1}$$

for $n \in \mathbb{Z}$. [2]
Here, z_n and a are parameters, with z_n being a 'time-dependent' parameter.

To simplify some of the considerations, we'll autonomise the system (setting $z_n = z$ for all n), and we'll introduce the change of parameters $\alpha = \frac{1}{2}(z + a)$ and $\beta = \frac{1}{2}(-z + a)$ to obtain the following system:

$$\begin{aligned} u_{n+1} + u_{n-1} &= \frac{(\alpha - \beta)u_n + \alpha + \beta}{1 - u_n^2} \\ &= \frac{\alpha(1 + u_n) + \beta(1 - u_n)}{1 - u_n^2} \\ &= \frac{\alpha}{1 - u_n} + \frac{\beta}{1 + u_n}. \end{aligned}$$

Setting $(x_n, y_n) = (u_n, u_{n-1})$, we obtain

$$x_{n+1} = \frac{\alpha}{1 - x_n} + \frac{\beta}{1 + x_n} - y_n, \tag{2}$$

$$y_{n+1} = x_n. \tag{3}$$

Thus, the autonomous $\mathrm{DP_{II}}$ equation is equivalent to iterating the following map $\mathbb{D} : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\mathbb{D} : \quad (x, y) \quad \mapsto \quad \left(\frac{\alpha}{1 - x} + \frac{\beta}{1 + x} - y, x \right).$$

2 $\mathrm{DP_{II}}$ as a QRT Mapping

2.1 The QRT Mapping

Given two matrices

$$A_i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \delta_i & \epsilon_i & \zeta_i \\ \kappa_i & \lambda_i & \mu_i \end{pmatrix}, \qquad i = 0, 1, \tag{4}$$

one defines two vector functions

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} := (A_0\mathbf{X}) \times (A_1\mathbf{X}) \tag{5}$$

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} := (A_0^T\mathbf{X}) \times (A_1^T\mathbf{X}), \tag{6}$$

where $\mathbf{X} := (x^2, x, 1)^T$. The Quispel-Roberts-Thompson mapping (QRT) associated to (A_0, A_1) is given by

$$x_{n+1} = \frac{f_1(y_n) - x_n f_2(y_n)}{f_2(y_n) - x_n f_3(y_n)} \tag{7}$$

$$y_{n+1} = \frac{g_1(x_{n+1}) - y_n g_2(x_{n+1})}{g_2(x_{n+1}) - y_n g_3(x_{n+1})}. \tag{8}$$

Theorem. (QRT Constant of Motion) The system (7)-(8) has an invariant (or contant of motion) for each orbit of the mapping:

$$\begin{aligned}(\alpha_0 + K\alpha_1)x_n^2y_n^2 + (\beta_0 + K\beta_1)x_n^2y_n + (\gamma_0 + K\gamma_1)x_n^2 + (\delta_0 + K\delta_1)x_ny_n^2 \\ + (\epsilon_0 + K\epsilon_1)x_ny_n + (\zeta_0 + K\zeta_1)x_n + (\kappa_0 + K\kappa_1)y_n^2 + (\lambda_0 + K\lambda_1)y_n + (\mu_0 + K\mu_1) = 0.\end{aligned} \tag{9}$$

Thus, being able to recognise a discrete dynamical system as QRT mapping has the immediate benefit of throwing out a constant of motion for free!

2.2 $\mathrm{DP_{II}}$ is a QRT Mapping

Taking

$$A_0 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & \beta - \alpha & -(\alpha + \beta) \\ 1 & -(\alpha + \beta) & \mu \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

yields the following for QRT:

$$F(x) = G(x) = \begin{pmatrix} (\beta - \alpha)x - (\alpha + \beta) \\ x^2 - 1 \\ 0 \end{pmatrix}$$

so that the associated QRT mapping is

$$x_{n+1} = \frac{\alpha}{1 - y_n} + \frac{\beta}{1 + y_n} - x_n = \mathbb{D}(y_n, x_n)$$

$$y_{n+1} = \frac{\alpha}{1 - x_n} + \frac{\beta}{1 + x_n} - y_n = \mathbb{D}(x_n, y_n).$$

Setting $u_{2n} = x_n$ and $u_{2n+1} = y_n$ recovers precisely the equation

$$u_{n+1} = \frac{\alpha}{1 - u_n} + \frac{\beta}{1 + u_n} - u_{n-1},$$

which is the first form of the autonomous $\mathrm{DP_{II}}$ equation we encountered.

Thus, we have the following constant of motion:

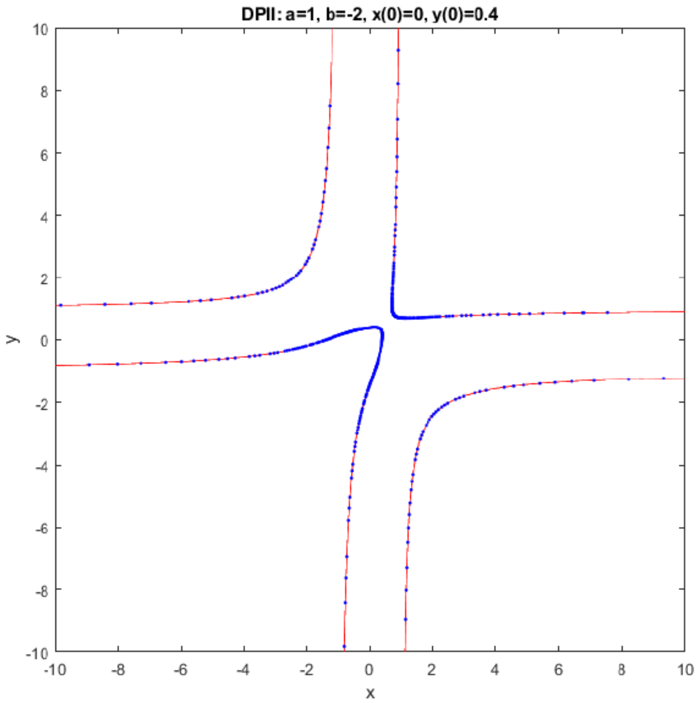
$$-x_n^2y_n^2 + x_n^2 + (\beta - \alpha)x_ny_n - (\alpha + \beta)x_n + y_n^2 - (\alpha + \beta)y_n + (\mu + K) = 0.$$

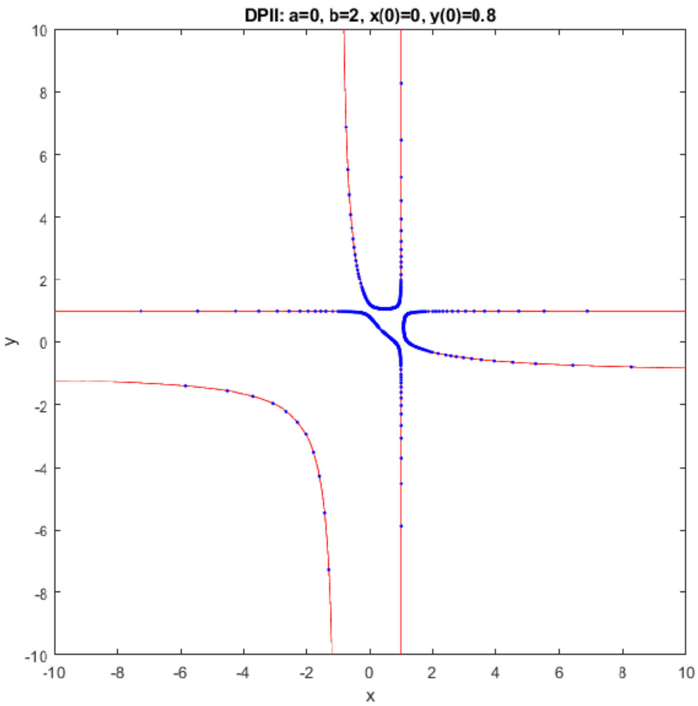
In particular, for the iterated map $\mathbb{D} : \mathbb{R}^2 \to \mathbb{R}^2$, we have that the following is constant on orbits

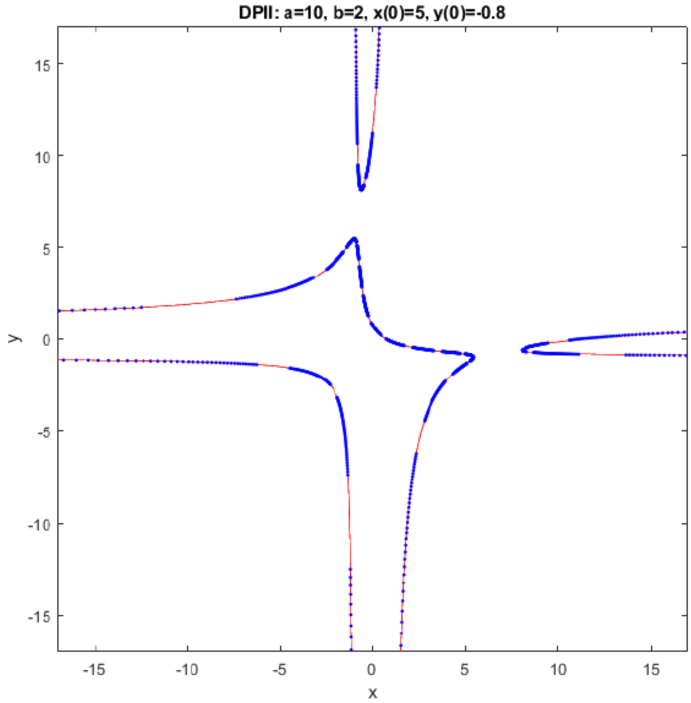
$$x^2y^2 - x^2 - y^2 + (\alpha - \beta)xy + (\alpha + \beta)(x + y). \tag{10}$$

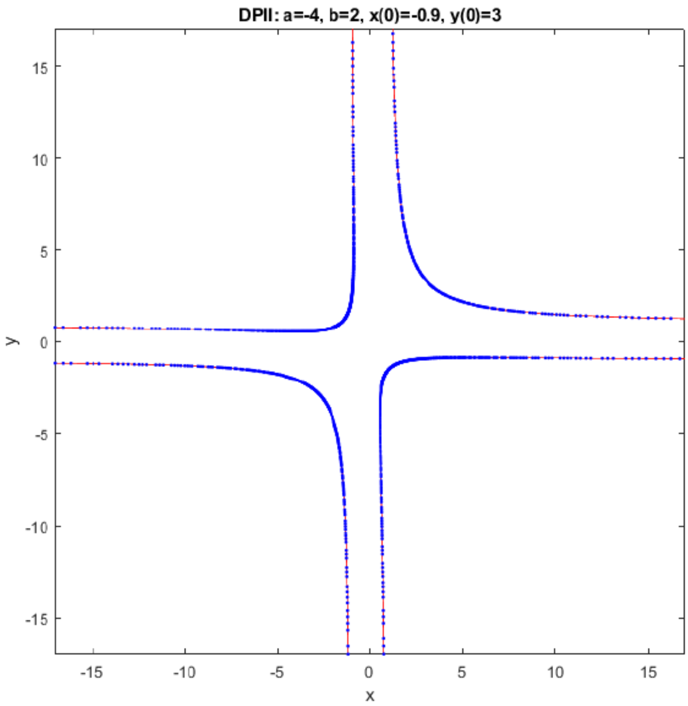
3 Orbits of DP_{II}

To motivate the next section, I've plotted some trajectories for DP_{II} on top of the corresponding constants of motions prescribed in the previous section.









In all of the above, the red curve is an implicit plot of the level set of the polynomial in Equation 10 corresponding to the given parameters and initial point.

The blue dots are the points of the orbit $\mathcal{O}(x_0, y_0) = \{\mathbb{D}^n(x_0, y_0) : \quad n \in \mathbb{N} \cup \{0\}\}$.

4 Projectivisation

In all of the above plots, we see that our orbits go 'off to infinity' in two ways: with slope zero or slope 'infinity'. We'll use projective space to extend the dynamics to infinity, so that two paths going to/coming from infinity, with the same slope, will meet. This lines up precisely with the notion of singularity confinement in [1].

Real projective 2-space, $\mathbb{P}_2(\mathbb{R})$, is the quotient manifold obtained from $\mathbb{R}^3 \setminus \{0\}$ modulo homothety ($(t,x,y) \sim (t',x',y')$ iff $\exists\,\lambda \in \mathbb{R}\setminus\{0\}$ for which $(t',x',y') = \lambda(t,x,y)$). We denote points of $\mathbb{P}_2(\mathbb{R}) = {}^{\mathbb{R}^3\setminus\{0\}}/_{\sim}$ by $[t:x:y]$, where not all of t, x and y are zero.

There are three typically chosen charts (each a copy of $\mathbb{R}^2$):

$$\begin{aligned} t\text{-chart}: &\ \{[1:x:y]:\ x,y\in\mathbb{R}\} \\ x\text{-chart}: &\ \{[t:1:y]:\ t,y\in\mathbb{R}\} \\ y\text{-chart}: &\ \{[t:x:1]:\ t,x\in\mathbb{R}\}. \end{aligned}$$

The 'line' $t=0$ (which gives a copy of $\mathbb{P}_1(\mathbb{R})$) is how we depict infinity in this picture: $t \neq 0$ gives us back the (x,y)-plane, and what's left is infinity. We call $t=0$ the projective line at infinity.

To extend a rational mapping on $(f_1,f_2):\mathbb{R}^2\to\mathbb{R}^2$ to one on $\mathbb{P}_2(\mathbb{R})$, we simply apply the mapping on the t-chart $[1:x:y]\mapsto[1:f_1(x,y):f_2(x,y)]$, clear denominators, and homogenise the polynomials using t (which is equal to one in the t-chart). Polynomials $P:\mathbb{R}^2\to\mathbb{R}$ can similarly be extended to polynomials $P:\mathbb{P}_1(\mathbb{R})\to\mathbb{R}$ via homogenisation.

4.1 Projectivisation of $\mathbb{D}$

I'll use $\mathbb{PD}$ to denote the projectivisation of $\mathbb{D}$. On the t-chart, our map needs to agree with

$$[1:x:y]\mapsto\left[1:\frac{\alpha}{1-x}+\frac{\beta}{1+x}-y:x\right]\equiv[1-x^2:\alpha(1+x)+\beta(1-x)-y(1-x^2):x(1-x^2)].$$

Thus, our map $\mathbb{PD}:\mathbb{P}_2(\mathbb{R})\to\mathbb{P}_2(\mathbb{R})$ is given by

$$\mathbb{PD}:\ [t:x:y]\ \mapsto\ [t^3-tx^2:(\alpha+\beta)t^3+(\alpha-\beta)t^2x-t^2y+yx^2:t^2x-x^3]. \tag{11}$$

There are a few different things we can do with $\mathbb{PD}$ to further study $\mathbb{D}$. But, for the sake of brevity, I'll choose to use it to answer just one natural question: what has this added to the dynamics?

Away from $t=0$, we're just in the (x,y)-plane, and we've changed nothing. But, what the dynamics on $t=0$? Plugging in $t=0$ to (11) yields:

$$\mathbb{PD}:\ [0:x:y]\ \mapsto\ [0:yx^2:-x^3]. \tag{12}$$

Immediately, this tells us that, if we start on $t=0$, we stay on $t=0$. Further, if x and y are nonzero, then the map is

$$[0:x:y]\mapsto[0:yx^2:-x^3]\sim[0:y:-x]\mapsto[0:-xy^2:-y^3]\sim[0:-x:-y]=[0:x:y]$$

so that orbits starting on $t=0$ (away from $xy=0$) are periodic of period two.

4.2 Projectivisation of the Constant of Motion

We can also projectivise the constant of motion. For each orbit, we have an invariant (10)

$$x^2y^2-x^2-y^2+(\alpha-\beta)xy+(\alpha+\beta)(x+y)-E=0$$

where E is some constant associated to the orbit. Homogenisation yields

$$x^2y^2-t^2x^2-t^2y^2+(\alpha-\beta)t^2xy+(\alpha+\beta)t^3(x+y)-Et^4=0.$$

This intersects the line at infinity when $t=0$, i.e.

$$x^2y^2=0.$$

So, the line at infinity is hit at precisely two points $[0 : 1 : 0]$ and $[0 : 0 : 1]$ (not $[0 : 0 : 0]$, since this is not in $\mathbb{P}_2(\mathbb{R})$). The point $[0 : 1 : 0]$ corresponds to going off to infinity vertically, and the point $[0 : 0 : 1]$ corresponds to going off to infinity horizontally. This confirms that our observation always holds.

An issue: If we fix our parameters α and β, then, no matter what initial conditions we choose, we'll not have a unique solution passing through the points $[0 : 0 : 1]$ and $[0 : 1 : 0]$. In the next section, we'll see how to extend our manifold further (along with the dynamics) in such a way that our 'energy', E, uniquely picks out a level set. This will give us the desired existence and uniqueness of solutions.

5 Blowing Up

The following construction will be formed for the origin in $\mathbb{R}^2$. Since $\mathbb{P}_2(\mathbb{R})$ is covered by copies of $\mathbb{R}^2$, the construction will extend to $\mathbb{P}_2(\mathbb{R})$ by choosing charts appropriately and performing translations. The following can be found (for $\mathbb{C}^2$) in [3].

We'll effectively glue a copy of $\mathbb{P}_1(\mathbb{R})$ to $\mathbb{R}^2$ through the origin, in a 'smooth' way. Namely, we define the blow up of $\mathbb{R}^2$ at $(0, 0)$ to be the closed set in $\mathbb{R}^2 \times \mathbb{P}_1(\mathbb{R})$ given by

$$X := \{((x_0, x_1), [y_0 : y_1]) \in \mathbb{R}^2 \times \mathbb{P}_1(\mathbb{R}) : \ x_0 y_1 = x_1 y_0\}.$$

This set comes equipped with a projection map $\pi : X \twoheadrightarrow \mathbb{R}^2$ which maps $((x_0, x_1), [y_0 : y_1]) \mapsto (x_0, x_1)$ (surjective since away from $(0, 0)$, we can just take $y_0 = x_1$ and $y_1 = x_0$, and, at $(0, 0)$, we can take $[y_0 : y_1]$ to be arbitrary).

In fact, for $x = (x_0, x_1) \neq (0, 0)$, $\pi^{-1}(x)$ is a singleton. And we see that $\pi : X \setminus \pi^{-1}(0) \cong \mathbb{R}^2 \setminus \{0\}$. We also have $\pi^{-1}((0, 0)) = \{((0, 0), [y_0 : y_1]) : \ [y_0, y_1] \in \mathbb{P}_1(\mathbb{R})\}$ which is a copy of $\mathbb{P}_1(\mathbb{R})$.

Thus, nothing changes away from the blow up point, but a projective line is introduced at the blow up point.

Intuition: We should think of this $\mathbb{P}_1(\mathbb{R})$ as encoding slopes of incoming trajectories. Two trajectories crossing $(0, 0)$ with different slopes will hit this projective line at different points in the blow up. If two trajectories hit $(0, 0)$ tangentially, then we may have to repeat the blowing up process multiples times.

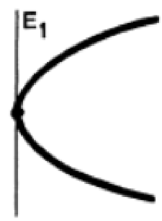
5.1 Example: The Cusp $(y^2 - x^3 = 0)$ [5]

Consider the variety defined by

$$y^2 - x^3 = 0.$$

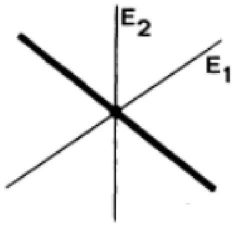
We'll introduce a projective line, with coordinates $[u, v]$, via the glueing $ux = vy$. In the chart where $v \neq 0$, we can divide by v to obtain $y = \frac{u}{v}x = wx$, where I'll use w to denote $\frac{u}{v}$. Thus

$$y^2 - x^3 = w^2 x^2 - x^3 = x^2(w^2 - x).$$

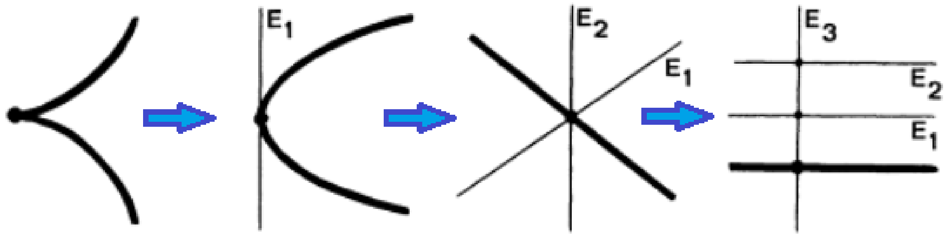


We need to blow up again (to remove the multiplicity):

$$x = tw, \; w = w: \qquad \Rightarrow \qquad w^2 - x = w^2 - tw = w(w-t).$$



We can blow this up one more time (to resolve the triple intersection), but I'll replace this step with a picture:



This last blow up is needed if we wish to have what is known as a strong desingularisation.

5.2 The $\mathrm{DP_{II}}$ Invariant

As the page limit for this extended abstract is close, I'll just perform the first blow up of the $\mathrm{DP_{II}}$ constant of motion. With each blow up, we see (as in the last example) that a factorisation drops the degree of the polynomial we're considering. Thus, since our invariant is of degree 4, it's clear that we'd need no more than 4 blow-ups in order for the energy E to distinguishable for different level sets at $t = 0$. In our case, we'll see that the first blow-up knocks off a whole two degrees in one fell swoop!

As we saw earlier, the invariant

$$x^2y^2 - t^2x^2 - t^2y^2 + (\alpha - \beta)t^2xy + (\alpha + \beta)t^3(x + y) - Et^4 = 0$$

has two projective solutions on $t = 0$, $[0 : 0 : 1]$ and $[0 : 1 : 0]$. I'll now blow up at the first point.
The point $[0 : 0 : 1]$ lives in the y-chart, where we can take $y = 1$. Making this substitution gives

$$x^2 - t^2x^2 - t^2 + (\alpha - \beta)t^2x + (\alpha + \beta)t^3(x + 1) - Et^4 = 0.$$

The point $[0 : 0 : 1]$ is identified with $(0, 0)$ in the (t, x)-plane, so we blow up there.
Setting $x = ut$, we obtain

$$0 = u^2t^2 - u^2t^4 - t^2 + (\alpha - \beta)ut^3 + (\alpha + \beta)t^3(ut + 1) - Et^4 = t^2(u^2 - u^2t^2 - 1 + (\alpha - \beta)ut + (\alpha + \beta)t(ut + 1) - Et^2).$$

So, the blown-up invariant becomes

$$u^2 - u^2t^2 - 1 + (\alpha - \beta)ut + (\alpha + \beta)t(ut + 1) - Et^2.$$

Of course, this needs to be blown up further (after a change of coordinates to 'recentre' the singularity to the origin), but this is left as a *fun* exercise for the reader!

References

[1] Basil Grammaticos, Thamizharasi Tamizhmani, Yvette Kosmann-Schwarzbach *Discrete Painlevé Equations: A Review*, Lecture Notes in Physics, 644, 245321 (2004)

[2] Masataka Kanki, Jun Mada, Tetsuji Tokihiro, *Discrete Painléve equations and discrete KdV equation over finite fields*, on arXiv.org (arXiv:1304.5039v2)

[3] Robin Hartshorne, *Algebraic Geometry*, Springer Verlag, ISBN 978-1-4419-2807-8, 1977.

[4] Tova Brown, *Asymptotics and Dynamics of Map Enumeration Problems (PhD Thesis)*, 2016.

[5] Wikipedia Page on Resolution of Singularities, `https://en.wikipedia.org/wiki/Resolution_of_singularities`, Accessed 12/11/2017.