Math 583B Spring 2012 Problem Set #7

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Due Wednesday, 5/2

Revised 2012.05.01

Exercises. *These do not need to be turned in.*

1. ~~Show that the identity operator I on the space $L^2([0,1])$ is not compact.~~ Show directly that the identity operator I on any infinite-dimensional Hilbert space is not the limit (in the metric defined by the operator norm) of a sequence of finite-rank operators. **rephrased**

2. Let $k : [0,1]^2 \to \mathbb{R}$ be a continuous kernel and define an integral operator K by

$$(Kf)(x) = \int_0^1 k(x,y)\, f(y)\, dy\ .$$

   (a) Let $X = C([0,1])$ denote the space of continuous functions on the interval $[0,1]$, equipped with the sup norm $||f|| = \max\{|f(x)| : 0 \leq x \leq 1\}$. Show that K is a bounded operator on X, and ~~compute the~~ give an upper bound for the operator norm $||K||$.

   (b) For a continuous f and any number λ with $|\lambda|$ sufficiently small, show that the Neumann series

$$u = \sum_{n=0}^{\infty} \lambda^n K^n f$$

   converges uniformly.

   (c) True or false: the Fredholm equation $u = f + \lambda K u$ has a finite dimensional solution space for all λ and all $f \in L^2([0,1])$? ~~(Assume that solutions u lie in L^2 .)~~ For this part, you should assume that K is compact.[1] **clarified + footnc**

3. ~~For any $f \in L^2(\mathbb{R})$ and real number $a \in \mathbb{R}$, define the operator T_a by $(T_a f)(x) = f(x-a)$. Is T_a compact for any a? Justify your answer.~~ For any $f \in L^2(\mathbb{R})$, define the operator T by $(Tf)(x) = f(-x)$. Give *two* different reasons why T is not compact. **old version not good; this is slightly more interesting**

4. For any vector field F on $\mathbb{R}^n$, find the differential equation satisfied by the path $x : [a,b] \to \mathbb{R}$ that minimizes

$$\int_a^b ||\dot{x}(t) - F(x(t))||^2\, dt$$

   where $||\cdot||$ denotes the standard Euclidean 2-norm. (This is known as the “Freidlin-Wentzell action” and arises in the theory of randomly-perturbed differential equations.)

[1] We know this is the case if K is viewed as an operator on $L^2([0,1])$. It is actually also true for K acting on continuous functions; the proof uses the Arzela-Ascoli theorem.

Problems.

1. The (idealized) spherical pendulum is a physical system consisting of a single particle of mass M constrained to be a fixed distance $\ell > 0$ from the origin; the mass is otherwise allowed to move freely. The configuration space is thus the 2-dim sphere of radius ℓ in $\mathbb{R}^3$, which can be parametrized by two angles (θ, φ) via spherical coordinates.

   This problem is about a spherical pendulum in a constant, downward gravitational field (i.e., generated by the usual potential $V(x, y, z) = gz$) .

   (a) Find a Lagrangian for this system.

   (b) Find the corresponding equations of motion in spherical coordinates.

   (c) What conserved quantities are implied by Noether's principle?

2. Our derivation of the Euler-Lagrange equations still works when there are > 1 independent variables, as is the case for spatially-dependent problems. This problem concerns a two-dimensional case.

   Let Ω be a bounded open subset of $\mathbb{R}^2$ and let $\partial\Omega$ denote the boundary of Ω . Define the functional

$$F[u] = \iint_\Omega L(x, y, u(x, y), u_x(x, y), u_y(x, y))\, dx\, dy$$

   where $u : \Omega \to \mathbb{R}$ is a smooth function. Our goal is to find the minimizer of F among all functions u subject to the Dirichlet boundary conditions

$$u(x, y) = f(x, y)\ , \quad (x, y) \in \partial\Omega\ ,$$

   where $f : \partial\Omega \to \mathbb{R}$ is a given (fixed) function.

   (a) To derive the Euler-Lagrange equations for higher-dimensional problems, you will need the following generalization of integration by parts: let F be a vector field on Ω and h a scalar-valued function that vanishes on $\partial\Omega$. Show that **NEW PART**

$$\begin{aligned}&\iint_\Omega \text{div}\Big[F(x, y) \cdot h(x, y)\Big]\, dx\, dy \\ &\quad = \iint_\Omega \text{div}F(x, y) \cdot h(x, y)\, dx\, dy + \iint_\Omega F(x, y) \cdot \nabla h(x, y)\, dx\, dy \\ &\quad = 0\ .\end{aligned}$$

   (b) Using the above, derive the Euler-Lagrange equation. Justify all your steps.

   (c) What PDE do you get for the Lagrangian

$$L(x, y, u, v, w) = v^2 + w^2\ ?$$

   (d) **(EXTRA CREDIT)** Let $\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$. Express the Euler-Lagrange PDE you found in (c) in polar coordinates. *Hint: This is different from the kind of change of variables we talked about in class: we are changing the* <u>*independent*</u> *variables here, not the dependent variables. This means there's a jacobian that arises when you change variables, which needs to be included as part of the "new" Lagrangian.* **now EC, + hint**

3. Let Ω be the unit disc in $\mathbb{R}^2$, and let h be a function defined on the unit circle $\partial\Omega$. Suppose u is the smooth function on Ω whose graph has minimal surface area among all functions that satisfy $u(x, y) = h(x, y)$ for $(x, y) \in \partial\Omega$. What equation does u satisfy?