# Math 583A Fall 2011 Problem Set #2

klin@math.arizona.edu

Due Tuesday, 9/20

1. This problem concerns the PDE $u_{tt} - u_{xx} = u(1-u)$ .

   (a) Is there a traveling wave solution that satisfies all 3 of the following conditions: (i) meets the boundary conditions $u(-\infty, t) = u(+\infty, t) = 0$ for all $t \geq 0$; (ii) is non-constant in $x$, and (iii) has a positive wave speed $c$ ? If so, for what range of $c$ does it exist?

   (b) There are two constant solutions for this equation, $u_* \equiv 0$ and $u_* \equiv 1$. ~~Discuss their linear stability,~~ Assuming periodic boundary conditions on the interval $[0, L]$, linearize the PDE about $u_*$ and solve the resulting linear PDE using Fourier series. How the $n$th mode behave as $L$ varies?

2. Consider

$$\begin{aligned} \dot{x} &= -y + ax(x^2 + y^2) \ , \\ \dot{y} &= \quad x + ay(x^2 + y^2) \ . \end{aligned}$$

   (a) Derive the linearization around the fixed point at the origin, and compute its eigenvalues and eigenvectors. Classify its behavior.

   (b) What is the behavior of the fully nonlinear system when (i) $a > 0$, (ii) $a = 0$, and (iii) $a < 0$? *Hint: Polar coordinates.*

3. Consider

$$\begin{aligned} \dot{x} &= xy \\ \dot{y} &= x^2 - y \ . \end{aligned}$$

   (a) Show that the linearization predicts that the origin is a non-isolated fixed point.

   (b) Show that the origin is in fact an isolated fixed point.

   (c) Is the origin repelling, attracting, a saddle, or something else? You can guess the answer by sketching the phase plane around the origin.[1] *Hint: You may find it helpful to first sketch the vector field along the* nullclines, *i.e., the curves along which* $\dot{x} = 0$ *and* $\dot{y} = 0$*. Note that the fixed point is, by definition, at the intersection of the nullclines!*

   (d) Use a computer to verify your answer. A useful tool is "pplane," which you can find at http://math.rice.edu/∼dfield/dfpp.html . You can also use any other tool you'd like. Please include a print-out of your phase plane.

4. (a) Verify the Cauchy-Riemann equations for $f(z) = z^2$ .

   (b) For what $z$ is the function $f(z) = (x + \alpha y)^2 + 2i(x - \alpha y)$ analytic? Here, $\alpha$ is a real constant, and as usual we write $z = x + iy$ for $x, y$ real.

5. Show that $f$ is analytic if and only if $g(z) = \overline{f(\overline{z})}$ is analytic.

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[1] Those of you who know about center manifolds should note that this problem can be approached via center manifold theory. You do not need to do that here — though of course you're welcome to take a shot at it!