Math 583A Fall 2011 Problem Set #3

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Due Tuesday, 10/4

Note: We haven't done all these types of problems in class, and not all of them are explained in detail in the notes. If you don't know how to start, please see me or Stuart. —KL

1. For each of the following functions, find all singularities and identify their type (removable, essential, or pole).

   (a) $\frac{e^{z^2}-1}{z^2}$

   (b) $\frac{e^{2z}-1}{z^2}$

   (c) $e^{\tan(z)}$

   (d) $\frac{z^3}{z^2+z+1}$

2. Let C be the simple closed contour defined by the square whose axes are parallel to the real and imaginary axes, with diagonal corners are $-1-i$ and $1+i$. Integrate the following functions around C:

   (a) $\bar{z}$

   (b) $\mathrm{Re}(z)$

3. Find all Taylor and Laurent series for f centered at a:

   (a) $f(z) = \frac{e^z}{(z-1)^4}\ , \quad a = 1$

   (b) $f(z) = \frac{1}{4+z^2}\ , \quad a = 0$

   (c) $f(z) = \frac{7z^2+9z-18}{z^3-9z}\ , \quad a = 0$

4. Let C denote the unit circle centered at 0. Evaluate the following contour integrals using residues.

   (a) $\int_C \frac{z+1}{2z^3-3z^2-2z}\, dz$

   (b) $\int_C \frac{\cosh(1/z)}{z}\, dz$

5. Evaluate using residues:

   (a) $\int_{-\infty}^{+\infty} \frac{\cos(kx)}{(x^2+1)(x^2+a^2)}\, dx\ ; \quad a^2, k > 0$

   (b) $\int_{-\infty}^{+\infty} \frac{xe^{ix}}{\cosh(x)}\, dx$ *Note: See Example 12 in Sect. 2.7 of the notes*

6. (a) Let f be entire, and let $M(R) = \max\{|f(z)| : |z| = R\}$. Prove that for all $R > 0$ and all $|z| < R$, the nth derivative of f satisfies

$$|f^{(n)}(z)| \leq n! \cdot \frac{R}{d(z, R)^{n+1}} \cdot M(R)$$

where $d(z, R)$ is the minimum distance from z to the circle $\{|z| = R\}$.

(b) Using your result from Part (a), prove that if a polynomial $P(z)$ has no roots, then it must be a constant. *Hint: First show that if P has no roots, then $f = 1/P$ must be a bounded entire function.*

(c) Let f and M be as in Part (a). Suppose $M(R) \leq a + bR^k$ for all $R \geq 0$, where $a, b > 0$ and k is a positive integer . Show that f must be a polynomial of degree $\leq k$.