Math 583A Fall 2011 Problem Set #5

klin@math.arizona.edu

Due **Thursday, 10/27**

Last revised: 2011.10.26

1. **Some basic properties of Fourier coefficients.** Let $f : \mathbb{R} \to \mathbb{C}$ be a periodic function of period 2π with continuous derivatives up to order k , and let $\hat{f}(n)$ denote the nth Fourier coefficient of f. Show

   (a) $\widehat{f^{(k)}}(n) = (in)^k \cdot \widehat{f}(n)$

   (b) $\widehat{\overline{f}}(n) = \overline{\widehat{f}(-n)}$

   (c) $\widehat{T_a f}(n) = e^{-ina}\widehat{f}(n)$, where $(T_a f)(x) = f(x-a)$

   (d) f is real-valued $\Leftrightarrow \widehat{f}(n) = \overline{\widehat{f}(-n)}$ for all n

   (e) f is real-valued and even $\Leftrightarrow \widehat{f}(n) \in \mathbb{R}$ and $\widehat{f}(n) = \widehat{f}(-n)$ for all n

   (f) f is real-valued and odd $\Leftrightarrow \widehat{f}(n) \in i\mathbb{R}$ and $\widehat{f}(n) = -\widehat{f}(-n)$ for all n

   Note: Each of these should involve only a short calculation (I'm not looking for rigorous proofs here). Also, for (a), you can start with the integral for $\widehat{f^{(k)}}$ or with the series — up to you which way.

2. Let $f : [-\pi, \pi] \to \mathbb{R}$ be the step function

$$f(x) = \begin{cases} -1, & x < 0 \\ +1, & x > 0 \end{cases}$$

   (a) Find the Fourier coefficients of f .

   (b) By applying the Parseval identity to f, show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} .$$

   (c) Using (b), show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} .$$

   (In case you haven't seen this before and are curious, $\sum 1/n^4 = \pi^4/90$. One way to calculate this sum is to repeat the above problem with $f(x) = |x|$, for which you can use Problem 1(a). With a little bit more work, you can find a recursion relation for $\sum 1/n^s$ for even positive integers s ; the sum is not known to have an explicit expression for any odd s.)