Math 583A Fall 2011 Problem Set #6

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Due **Tuesday, 11/15**

Last revised: 2011.11.14
($\star$ = corrected problems)

1. ($\star$) *A linear inhomogeneous PDE.* Find the Fourier series solution of the 2D PDE

$$u_{tt} = c^2(u_{xx} + u_{yy}) + \sin(y) , \quad 0 \le x \le 1, \quad 0 \le y \le \pi ,$$

with homogeneous Dirichlet boundary conditions $u \equiv 0$ for $y \in \{0, \pi\}$, homogeneous Neumann conditions $u_x \equiv 0$ for $x \in \{0, 1\}$, and initial conditions $u(x, y, 0) = \cos(\pi x) \sin(3y)$ and $u_t(x, y, 0) = 0$. *Hint: It's easier to first think about how to match the boundary conditions term by term.*

2. *Fourier representation of a nonlinear PDE.* Consider the PDE

$$u_t + uu_x = \nu u_{xx} , \qquad u : [-\pi, \pi] \times \mathbb{R} \to \mathbb{R} \text{ with periodic boundaries.}$$

Let $\widehat{u}_n(t)$ denote the nth Fourier coefficient of $u(x, t)$ as a function of x , i.e.,

$$\widehat{u}_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x, t) e^{-inx} \, dx .$$

Find a system of (infinitely many) coupled ODEs for $\widehat{u}_n(t)$.

3. *Using complex variables to find Fourier series.* There is another situation where Fourier series converge nicely (i.e., uniformly), namely when the function being expanded is analytic. Let $F : \mathbb{C} \to \mathbb{C}$ be analytic except for a countable number of singularities, none of which are on the circle $\{|z| = 1\}$. Then $f(\theta) = F(e^{i\theta})$ is clearly 2π-periodic and smooth.

   (a) Suppose F has Laurent expansion $F(z) = \sum_{n=-\infty}^{\infty} C_n z^n$. Find $\widehat{f}(n)$.

   (b) Compute the Fourier coefficients for

$$f(\theta) = \frac{1}{2 - \cos(\theta)} .$$

   Hint: Let $z = e^{i\theta}$ and try to rewrite f as an analytic function in z .

4. ($\star$) Using Cauchy-Schwartz, show that there is a constant $C > 0$ such that for all continuously-differentiable 2π-periodic $f : \mathbb{R} \to \mathbb{C}$,

$$|f(x) - \overline{f}| \le C||f'||_{L^2([-\pi,\pi])} ; \quad \overline{f} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx ,$$

for all x . (This problem illustrates a general principle that is often useful, namely that knowing the size of the derivative in an average (e.g., L^2) sense lets one obtain point-wise bounds on the value of a function.)

5. ($\star$) *Gibbs phenomenon.* Let $f : \mathbb{R} \to \mathbb{R}$ be the periodic extension (with period 2π) of the step function

$$f(x) = \begin{cases} -1, & x < 0 \\ +1, & x > 0 \end{cases}$$

(a) Show that

$$\lim_{N\to\infty}\Big[f(x_{N,k}) - (S_N f)(x_{N,k})\Big] = 2 - \frac{2}{\pi}\int_{-\infty}^{\pi k} \frac{\sin(u)}{u}\, du\ , \tag{1}$$

where $x_{N,k} = 2k\pi/(2N+1)$.

(Note: I write the upper limit as $4k\pi$ before. Stupid error on my part.)

(b) Evaluate[1] Eq. (1) for $k = 1$, and compare it against a graph of the partial sum $S_N f$ for $N = 10$.

[1] Any way you like, including numerical.