Math 583A Fall 2011 Problem Set #7

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Due ~~Tuesday, 11/29~~ **Thursday, 12/1**

Last revised: 2011.11.30

$\star$ Read Sect. 4.6
$\star\star$ Revised

1. ($\star\star$) Consider the *Schrödinger equation*

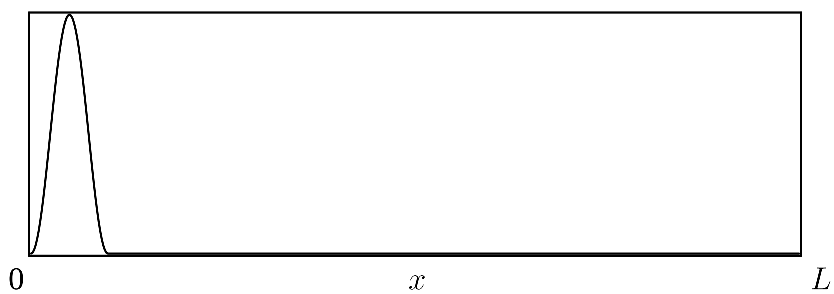
$$i\hbar u_t = -\frac{\hbar^2}{2m} u_{xx} \;, \tag{1}$$

with periodic boundary conditions on $[0, L]$ and initial condition $u(x, 0) = u_0(x)$. It describes the motion of a free particle of mass $m > 0$ moving around a circle. The "wave function" u is complex-valued, and $\hbar > 0$ is *Planck's constant.*

   (a) Find the dispersion relation for this equation.

   (b) Suppose $u_0(x) = e^{ikx}$. For what values of k is the result a traveling wave, and what is the corresponding wave speed?

   (c) ($\star\star$) Now let φ be a C^∞ function with compact support of length a, looking like this:



Suppose $u_0(x) = e^{ikx}\varphi(x)$ and $1/k \ll a \ll L$. (The symbol "$\ll$" means "much smaller than"; you can assume k here satisfies whatever condition you find in (b).) For small $t > 0$, which way, and how fast, do you expect the wave packet to move?
Note: I rewrote this problem so the various physical assumptions are more clear.

2. ($\star\star$) Let g be a Schwartz function and $0 < a < 1$. Find explicit expressions for

   (a)

$$\Big\langle \delta\big(a - 4x(1-x)\big),\ g(x)\Big\rangle$$

   (b) **(Extra Credit)** ($\star\star$)

$$\Big\langle \delta'\big(a - 4x(1-x)\big),\ g(x)\Big\rangle$$

where δ' is the derivative of the δ distribution.
Note: In expressions like $\delta'(f(x))$, differentiation takes precedence over composition. So you should interpret the above as saying $\langle\delta'(f(x)), g(x)\rangle = \langle\delta'(x), L_f g(x)\rangle$, where $L_f g$ is ~~the linear operator you found in (a).~~ is the linear operator defined implicitly via change of variables, i.e., $\langle\varphi(f(x)), g(x)\rangle = \langle\varphi(x), L_f g(x)\rangle$ for any generalized function φ (including δ and δ').

3. ($\star\star$) Find a solution g to the equation

$$g'(x) + ixg(x) = \delta(x) \ .$$

Hint: Look for an integrating factor.

Note: I added a factor of i to the above so that we can continue to use tempered distributions. Otherwise, you will need to consider smaller spaces of test functions, e.g., the space of smooth functions with compact support. Note that this is the case whether you have $xg(x)$ or $-xg(x)$ in the equation above...

4. Consider the following sum

$$\sum_{n=0}^{\infty} \cos(nx)$$

viewed as a distribution on the space of test functions $C^\infty([-\pi, \pi])$. Let f be a test function. Compute $\langle \phi, f \rangle$, where ϕ denotes the generalized function defined by the sum above.

5. Consider the periodic function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{9/2}} \cdot \sin(n^2 x) \ .$$

   (a) What is the rate of decay of the Fourier coefficients of f? That is, if $f(x) = \sum_n c_n e^{inx}$, find $C, \beta > 0$ so that $|c_n| \leq Cn^{-\beta}$.

   (b) Is f continuous, and does its Fourier series converge uniformly?

   (c) Is f continuously differentiable, and does the Fourier series of f' converge uniformly?

   (d) Is f *twice* continuously differentiable? *Hint: Continuous functions on $[-\pi, \pi]$ are in $L^2([-\pi, \pi])$.*

(For (b), (c), and (d), please justify your answer by stating the relevant theorem(s) or providing a short proof.)