Math 583B Spring 2012 Problem Set #1

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Due Wednesday, 2/1

Exercises. *These do not need to be turned in.*

1. Compute the Fourier transforms of the following:
   (a) $f(x_1, \cdots, x_d) = e^{-(|x_1|+\cdots+|x_d|)}$
   (b) $f(x) = \cos(x)\ g(x)$; express your answer in terms of $\widehat{g}$
   (c) $f(x) = \sin(x)\ g(x)$; express your answer in terms of $\widehat{g}$
   (d) $f(x, y) = \cos(x)\sin(y)g(x, y)$; express your answer in terms of $\widehat{g}$ *(note correction)*
   (e) $f(x, y, z) = \cos(x)\sin(y)e^{-z^2}$
   (f) $f(x, y) = \frac{\partial^2}{\partial x \partial y}\big[\cos(x + y)e^{-(x^2+y^2)/2}\big]$
   (g) $f(x, y, z) = \begin{cases} 1, & (x, y, z) \in [-1, 1]^3 \\ 0, & \text{otherwise} \end{cases}$

2. For each of the following PDEs, find the differential equation (in t) satisfied by the spatial Fourier transform $\widehat{u}(\omega, t)$ of u. Note that you do *not* need to solve the differential equation.
   (a)
   $$u_t = (A + B\cos^2(\lambda x))u_{xx}$$
   with $u = u(x, t)$, $(x, t) \in \mathbb{R} \times [0, \infty)$, and $A, B, \lambda > 0$
   (b)
   $$u_{tt} = (A + B\cos^2(\lambda x)\sin^2(\eta y))\Delta u$$
   with $u = u(x, y, t)$, $(x, y, t) \in \mathbb{R}^2 \times [0, \infty)$, and $A, B, \lambda, \eta > 0$
   (c)
   $$u_t - u_x + 2u_y = e^{-x^2/2b^2}\cos(ay)$$
   with $u = u(x, y, t)$, $(x, y, t) \in \mathbb{R}^2 \times [0, \infty)$, and $a, b > 0$

Problems. *Please write these up and turn them in.*

1. Consider the PDE
   $$u(x) + \Delta^2 u(x) = f(x) ,$$
   where Δ denotes the Laplacian operator (and Δ^2 means applying the Laplacian *twice*), $f : \mathbb{R}^d \to \mathbb{R}$ is given, and $u : \mathbb{R}^d \to \mathbb{R}$ satisfies the boundary conditions $u(x) \to 0$ as $|x| \to \infty$. You can assume that f is sufficiently nice that its Fourier transform is well-defined.
   (a) Find $\widehat{u}(\omega)$ in terms of $\widehat{f}(\omega)$.
   (b) Suppose $f(x) = \delta(x)$ and $d = 1$. Is u continuously-differentiable as a function of x?

(c) Suppose $f(x) = \delta(x)$. For what values of d do the functions u , $\frac{\partial u}{\partial x_1}, \cdots , \frac{\partial u}{\partial x_d}$ simultaneously belong to $L^2(\mathbb{R}^d)$?

2. Consider the wave equation in 3-D, i.e.

$$u_{tt} = \Delta u \ , \quad u : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R} \ ,$$

with initial conditions

$$\begin{aligned} u(x, 0) &= 0 \\ u_t(x, 0) &= \delta(x) \ . \end{aligned}$$

Assume that $u(x, t)$ decays sufficiently rapidly as $|x| \to \infty$ so that its Fourier transform is well-defined.

(a) Find the Fourier representation of the solution, i.e., $\widehat{u}(\omega, t)$, $\quad \omega \in \mathbb{R}^3$.

(b) Transform to spherical polar coordinates and show that

$$u(r, t) = \frac{1}{4\pi^2 r} \int_0^\infty \big[\cos(\rho(r - t)) - \cos(\rho(r + t)) \big] \, d\rho \ ,$$

where $r = |x|$ and $\rho = |\omega|$.