Math 583B Spring 2012 Problem Set #5

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Due Friday, 4/6
Revised 4/3

Read Sects. 7.1 - 7.4 of the course notes.

Exercises. *These do not need to be turned in.*

1. Find the Green's function for

$$-u''(x) + u(x) = f(x)\ ,\ \ 0 \le x \le 1$$
$$u(0) - u'(0) = 0$$
$$u(1) = 0\ .$$

2. One thing we have not discussed is what to do when there are nonzero (i.e., inhomogeneous) boundary conditions, i.e., when we want to solve $Lu = f$ on $0 \le x \le \ell$, where

$$Lu = (p(x)u'(x))' + q(x)u(x)$$

and

$$\alpha_1 u + \alpha_2 u' = A \text{ at } x = 0$$
$$\beta_1 u + \beta_2 u' = B \text{ at } x = \ell$$

where $(A, B) \ne (0, 0)$. The standard method is to split the solution into two terms: a *particular solution* u_p solving $Lu_p = f$ with the boundary conditions $A = B = 0$, and a *homogeneous solution* u_h solving $Lu_h = 0$ with the given nonzero boundary conditions. By linearity, $u = u_h + u_p$ will solve the original problem.

(a) Suppose the boundary value problem above has a unique solution for every f, **clarified** and let K denote the Green's function for L with homogeneous boundary conditions. Assume also the boundary conditions are such that $u(0), u(\ell) \ne 0$. **correction** Show, using the general properties of Green's functions, that every homogeneous solution can be expressed in the form

$$u_h(x) = c_1 K(x, 0) + c_2 K(x, \ell)\ .$$

Note: Since our general method for finding K relies on solving $Lu = 0$, this is really only useful if one already knows K by some other means.

(b) Solve, using whatever method is convenient,

$$-u''(x) + u(x) = \cos(\pi x)\ ,\ \ 0 \le x \le 1$$
$$u(0) - u'(0) = 1$$
$$u(1) = 0\ .$$

Problems.

1. The method we used in class to derive a general expression for the Green's function for the Sturm-Liouville problem can be directly applied (i.e., without doing a Sturm-Liouville reduction) to general second-order problems of the form $Lu = f$, where

$$Lu(x) = p_2(x)u''(x) + p_1(x)u'(x) + p_0(x)u(x)\ , \quad 0 \le x \le \ell$$
$$\alpha_1 u + \alpha_2 u' = 0 \text{ at } x = 0$$
$$\beta_1 u + \beta_2 u' = 0 \text{ at } x = \ell$$

Let K be the Green's function for L. You can assume the p_i are continuous, and $p_2(x) > 0$ for all $x \in [0, \ell]$.

(a) What equations (including boundary conditions) does K satisfy?

(b) Derive a jump condition for K.

(c) Find a general expression for K in terms of two linearly indepenent solutions u_1 and u_2 of $Lu = 0$.

(d) Find the Green's function for the BVP

$$u''(x) - 2u'(x) + u(x) = f(x)\ , \ \ 0 \le x \le 1$$
$$u(0) = u(1) = 0\ .$$

and use it to find an expression for u when $f \equiv 1$.

2. A self-adjoint operator L is *positive* if $\langle Lu, u \rangle > 0$ for all u in the domain of L with $||u|| > 0$.

(a) Let L be the Sturm-Liouville operator

$$Lu = -\big(p(x)u'(x)\big)' + q(x)\ u(x)\ , \quad 0 \le x \le \ell$$

acting on the space of functions satisfying ~~the usual boundary conditions~~ Dirichlet **correction** boundary conditions

$$u(0) = u(\ell) = 0$$

and with the standard inner product $\langle u, v \rangle = \int_0^\ell u(x)v(x)\ dx$. Show that if $q(x) > 0$ for all x, then L is positive.

(b) Let K denote the Green's function for L, and suppose L is positive. Does it follow that $K(x, y) > 0$ for all $x, y \in (0, \ell)$? Explain.

3. (a) Let $u : \mathbb{R}^n \to \mathbb{R}$ and define the Laplacian operator Δ by

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}\ .$$

Find the Green's function for Δ for all $n \ge 3$, assuming vanishing boundary conditions at ∞ .

(b) Find the Green's function for Δ on the half space $\mathbb{R}^2 \times [0, \infty)$.