The analogy between Fourier series representations and coordinates relative to orthogonal bases suggests various interesting and useful identities. Here is one such identity. Please write up #1 & 2 and turn them in. #3 is optional: do it if you are curious, and whenever you have time.

1. Let $x$ be any vector in $\mathbb{C}^n$ and $(v_1, \ldots, v_n)$ be an orthogonal basis of $\mathbb{C}^n$ with the property that $||v_n|| = 2$ for all $n$. Let $(c_1, \ldots, c_n)$ denote the coordinates of $x$ relative to the basis $(v_1, \ldots, v_n)$. Show that $||x||^2 = K \sum_{k=1}^{n} |c_k|^2$ for some constant $K$. What is the value of $K$?

   Hint: $||x||^2 = (x, x)$; use this and the orthogonality of the basis.

2. Let $f$ and $g$ be two functions, and define (as we did in class)

   $$(f, g) = \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx.$$  (1)

   Define $||f|| = \sqrt{(f, f)}$, and also

   $$v_n(x) = e^{inx}.$$  (2)

   As we discussed in class on Monday, the functions $v_n$ are “orthogonal” to each other in the sense that

   $$(v_m, v_n) = \int_{-\pi}^{\pi} e^{imx} \overline{e^{inx}} \, dx
   = \int_{-\pi}^{\pi} e^{inx} e^{-inx} \, dx
   = \left\{ \begin{array}{cl} 0, & m \neq n \\ 2\pi, & m = n \end{array} \right.$$

   Using this, show that

   $$||f||^2 = K \sum_{n=-\infty}^{\infty} |c_n|^2$$  (3)

   for some constant $K$. What is the value of $K$?

   (In many physical problems, $||f||^2$ can be interpreted as the energy per unit time contained in a wave, and Eq. (3) shows that the energy can be expressed in terms of the Fourier coefficients directly.)

3. Let $f$ be the function

   $$f(x) = \left\{ \begin{array}{cl} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{array} \right.$$  (4)

   We showed in class that $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ with

   $$c_n = \left\{ \begin{array}{cl} 0, & n \text{ even} \\ \frac{2i\pi n}{\pi^2}, & n \text{ odd} \end{array} \right.$$  (5)

   Apply the two sides of Eq. (3) to the Fourier series $f(x)$ above to find an explicit expression for $\sum_{n=-\infty}^{\infty} |c_n|^2$. 