These are practice problems for Fourier transforms. I will not collect these, but will post solutions before the final.

Note the first one is the example from class today (Monday 5/4).

1. Let \( g \) be the function

\[
g(x) = \begin{cases} 
1, & 0 \leq x \leq 1 \\
-1, & -1 \leq x \leq 0 \\
0, & \text{otherwise}
\end{cases}
\]

(a) Sketch \( g \).

(b) Let

\[
f(x) = \begin{cases} 
1, & -1 \leq x \leq 1 \\
0, & \text{otherwise}
\end{cases}
\]

Express \( g \) in terms of \( f \) via scaling, shifting, and linear combinations. Hint: let \( g_+ \) be the “right half” of \( g \), and \( g_- \) the left half. Show that each of \( g_+ \) and \( g_- \) can be obtained from \( f \) by scaling, then shifting. Or shifting, then scaling – you can do it in either order.\(^1\)

(c) Find \( \hat{g}(\omega) \).

(d) Try to do the same for

\[
g(x) = \begin{cases} 
2, & 1 \leq x \leq 2 \\
-3, & -1 \leq x \leq 0 \\
0, & \text{otherwise}
\end{cases}
\]

2. Consider the differential equation

\[-u''(x) + u(x) = f(x)\]

where \(-\infty < x < \infty\) and we assume \( u, f \) have well-defined Fourier transforms.

(a) Find \( \hat{u}(\omega) \) in terms of \( \hat{f}(\omega) \).

(b) By taking inverse Fourier transforms, express \( u(x) \) as a convolution \((K * f)(x)\). What is the function \( K \)? Hint: see the solution to Problem 4 on Exam 3.

\(^1\)If you scale, then shift, you would get \( g_+(x) = f(2(2 - \frac{1}{2})) \). If you shift, then scale, you get \( g_+(x) = f(2x - 1) \).

\(^2\)As an example, if you get \( g_+ \) from \( f \) by shifting then scaling, you would have \( g_+(x) = f(2x - 1) \). The two operations can be represented symbolically as

\[g_+ = \text{Scale(Shift}(f)).\]

From this, we have

\[
\mathcal{F}(g_+) (\omega) = \mathcal{F} (\text{Scale(Shift}(f)))) (\omega) \\
= \frac{1}{2} \mathcal{F} (\text{Shift}(f))(\omega/2) \\
= \frac{1}{2} e^{-i\omega/2} \mathcal{F}(f)(\omega/2).
\]
3. The Fourier transform of

\[ f(x) = \sqrt{\frac{2}{\pi}} \frac{\sin(x)}{x} \]

is

\[ \hat{f}(\omega) = \begin{cases} 1, & -1 \leq \omega \leq 1 \\ 0, & \text{otherwise} \end{cases} \]

(You can derive this by noticing that the Fourier transform of \( f \) is the same as its inverse Fourier transform, because \( f \) is even.) Assuming this fact, answer the following questions:

(a) Let \( g(x) \) be a function such that \( \hat{g}(\omega) = 0 \) for \( |\omega| < 1 \). What can you say about the Fourier transform of \( f \ast g \)?

(b) Let \( g(x) \) be a function such that \( \hat{g}(\omega) = 0 \) for \( |\omega| > 1 \). What can you say about the Fourier transform of \( f \ast g \)?