GEOMETRY OF TIME-DELAY COORDINATES

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Abstract: Under certain conditions we can reconstruct an attractor using time delay maps. This project describes a theorem by Berry, Cressman, Gregurik-Ferencec and Sauer that allows us to use a time delay embedding to reconstruct said dynamics

1. Introduction

According to Strogatz, Roux(1983) exploited a data analysis technique known as attractor reconstruction. The dynamics in the full phase space can be reconstructed from measurements from a single time series. The method is based in time delays. How can we choose the embedding dimension? One needs enough delays so that the attractor can disentangle itself in phase space. the following example, taken from Strogatz uses one delay so that $x(t) = (B(t), B(t-\tau))$, $\tau = 8.8$ Figure1 one ahows the time series and Figure 2 show the results of this reconstruction.

Conventional techniques of dimensionality reduction such as Karhunen-Loeve decomposition have been unable to recover a low-dimensional process. A key feature of delay coordinates is that they project the data onto the most stable dynamical variables. We're interested in separating time scales in the data by projecting into the most stable dynamical directions. It's important to separate the extrinsic features of the data, such as

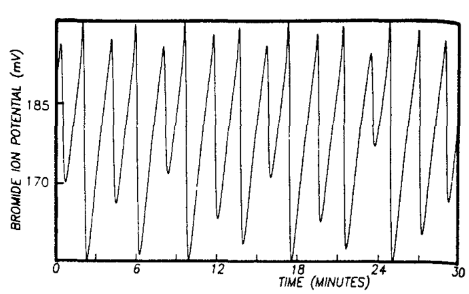

Figure 1. Time series

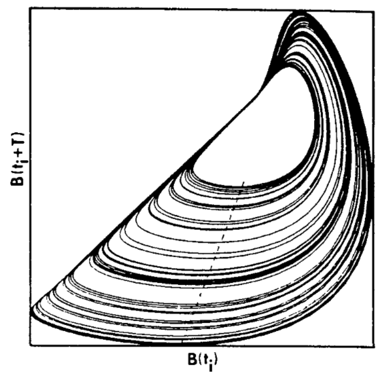

FIGURE 2. Attractor reconstruction

the observation and embedding space, from the intrinsic dynamical features such, as Lyapunov exponents and associated invariant manifolds. The fact that the embedding is given by a diffeomorphism of the attractor shows that the time-delay embedding is topology-preserving, although crucially it introduces a new geometry on the data set. We will see that the new geometry is related to the Lyapunov metric restricted to the most stable Oseledets subspace.

2. The theorem

Let M be an n-dimensional smooth compact Riemannian manifold which is the attractor of a system denoted $\dot{x} = f(x)$, with invariant measure μ for the induced flow F_t (fixed time set $\tau > 0$). According to Oseledets there exist real numbers $\sigma_1 \leq \cdots \leq \sigma_k$ with $k \leq n$ such that for μ-almost every x, there is a splitting $T_xM = \bigoplus_{i=1}^{k} E_i(x)$ where $\dim E_i = d_i$ and where $d_1 + \cdots + d_k = n$. Each Oseledets space $E_i(x)$ is invariant under the dynamics, meaning that any nonzero vector $u_i \in E_i(x)$ has image $DF_{-j\tau}(x)u_i \in E_i(DF_{-j\tau}(x))$. Moreover, for any $u_{i,x} \in E_i(x)$,

$$\lim_{j \to \infty} \frac{1}{j} \ln ||DF_{j\tau}(x)u_i|| = \sigma_i$$

$$\lim_{j \to -\infty} \frac{1}{j} \ln ||DF_{j\tau}(x)u_i|| = -\sigma_i$$

Assume a multivariate observation of dimension r, given by a smooth nonlinear $h \in C^{\infty}(M, \mathbb{R}^r)$. For $\kappa, \tau > 0$ define the κ-weighted delay coordinate map $H : M \to \mathbb{R}^{r(s+1)}$ by

$$H(x) = \left[h(x), e^{-\kappa}h(F_{-\tau}(x)), e^{-2\kappa}h(F_{-2\tau}(x)), \ldots, e^{-s\kappa}h(F_{-s\tau}(x))\right]^T$$

For each $\epsilon > 0$ the ϵ-Lyapunov metric is defined by $\langle u, v\rangle_\epsilon$ is defined by

$$\langle u_i, v_i\rangle_\epsilon = \sum_{j\in\mathbb{Z}} e^{-2(\sigma_i j+\epsilon|j|)}\langle DF_{j\tau}(x)u_i, DF_{j\tau}(x)v_i\rangle_{T_xM}$$

for $u_i, v_i \in E_i(x)$ The Lyapunov metric is intrinsic to the dynamics, and so it will be the Remannian metric of interest on M. Let's investigate the metric induced on the embedded manifold $H(M)$. Let $u = u_1 + \cdots + u_k, v = v_1 + \cdots + v_k \in T_xM$ where $u_i, v_i \in E_i(x)$ and denote $\hat{u} = DH(u), \hat{v} = DH(v) \in T_{H(x)}H(M)$. Then $\langle\cdot,\cdot\rangle$ in the Euclidean reconstruction space $\mathbb{R}^{r(s+1)}$ is

$$\langle\hat{u}, \hat{v}\rangle = \sum_{j=0}^{s} e^{-2j\kappa}\langle Dh(F_{-j\tau}(x))DF_{-j\tau}(x)u, Dh(F_{-j\tau}(x))DF_{-j\tau}(x))v\rangle_{\mathbb{R}^r}$$

The main theorem shows that, with the right choice of κ, the metric in the embedding space projects onto the most stable Oseledets subspace

Theorem

Let M be a compact manifold, $u, v \in T_xM$, and let $\hat{u} = DH(u)$ and $\hat{v} = DH(v)$ be the images under the time delay embedding H. Let $u_i = \pi_i(u)$ be the projection onto the ith Oseledets space, and assume that u_1 and v_1 are nonzero. Let $0 < \kappa < -\sigma_1$. Then for a prevalent choice of H and for all $i \neq 1$,

$$\lim_{s\to\infty} \frac{\langle\hat{u}_i, \hat{v}_i\rangle}{||\hat{u}||\,||\hat{v}||} = 0$$

and therefore,

$$\lim_{s\to\infty} \frac{\langle\hat{u}, \hat{v}\rangle - \langle\hat{u}_1, \hat{v}_1\rangle}{||\hat{u}||\,||\hat{v}||} = 0$$

We need to show that the component of $\hat{u}$ in the most stable direction dominates as s increases. Thus we will bound $||\hat{u}||$ from below, and we will focus on the component u_1 which is assumed to be nonzero. Choose $\kappa \geq 0$ such that $r(\kappa + 1) \geq n$; then the delay coordinate map is an immersion for a prevalent choice of H. From this we need only the fact that for all $x \in M$ and all $j \geq 0$ the rank of the $r(\kappa + 1) \times n$ matrix

$$A_j(x) = \begin{bmatrix} e^{-\kappa j}Dh(F_{-j\tau}(x))DF_{-j\tau}(x)) \\ \vdots \\ e^{-\kappa(k+j)}Dh(F_{-(k+j)\tau}(x))DF_{-(k+j)\tau}(x)) \end{bmatrix}$$

is n, implying that the kernel of the matrix is zero. For any nonzero vector $u_1 \in E_1(x)$ the vector $A_j(x)u_1$ is nonzero, and for some $j \leq l \leq j+k$, the r-vector $e^{-\kappa l}Dh(F_{-l\tau}(x))DF_{-l\tau}(x)u_1 \neq$

0. Thus for each $x \in M$ we have $\max_{j\leq l\leq j+k} ||Dh(F_{-l\tau}(x))e_1||_{\mathbb{R}^r} > 0$ for any unit vector $e_1 \in E_1(x)$. Since M is compact we can define

$$\begin{aligned} h_{\min} &\equiv \min_{x\in M} \min_{j\geq 0} \max_{j\leq l\leq j+k} ||Dh(F_{-l\tau}(x))e_1||_{\mathbb{R}^r} \\ &= \min_{x\in M, j\geq 0, y=F_{-j\tau}(x)} \max_{0\leq l\leq k} ||Dh(F_{-l\tau}(y))e_1||_{\mathbb{R}^r} \\ &\geq \min_{x\in M} \max_{0\leq l\leq k} ||Dh(F_{-l\tau}(x))e_1||_{\mathbb{R}^r} > 0. \end{aligned}$$

From this we get the lower bound

$$||A_j(x)u_1||_{\mathbb{R}^{r(k+1)}} \geq h_{\min} e^{-j(\kappa+\sigma_1+\epsilon)} \, ||u_1||_\epsilon$$

for all $x \in M$. We can use this to obtain a lower bound on $||\hat{u}||$ by splitting the s terms into $\lfloor s/k \rfloor$ block of size k so that

$$\begin{aligned} ||\hat{u}||^2 &\geq \sum_{j=1}^{s} e^{-2j\kappa} \, ||Dh(F_{-j\tau}(x))DF_{-j\tau}(x)u_1||_{\mathbb{R}^r}^2 \\ &\geq \sum_{l=0}^{\lfloor s/k \rfloor} ||A_{lk}(x)u_1||^2 \\ &\geq h_{\min}^2 \, ||u_1||_\epsilon^2 \, e^{-2k\lfloor s/k \rfloor(\sigma_1+\kappa+\epsilon)} \\ &\geq h_{\min}^2 \, ||u_1||_\epsilon^2 \, e^{-2ks(\sigma_1+\kappa+\epsilon)} \end{aligned}$$

By combining the upper and lower bounds, we have

$$\frac{\langle \hat{u}_i, \hat{v}_i \rangle}{||\hat{u}|| \, ||\hat{v}||} \leq \left(\frac{h_{\max}^2}{h_{\min}^2} \frac{||u_i||_\epsilon}{||u_1||_\epsilon} \frac{||v_i||_\epsilon}{||v_1||_\epsilon} \right) \frac{\left|1 - e^{-2(s+1)(\sigma_i+\kappa-\epsilon}\right|}{e^{-2s(\sigma_1+\kappa+\epsilon)}} \to 0$$

as $s \to \infty$ for $i \neq 1$. For large s, $\langle \hat{u}, \hat{v} \rangle \approx \langle \hat{u_1}, \hat{v_1} \rangle$ so the metric is negligible in all but the most stable Lyapunov direction.

3. Discussion

The proof fails if the hypothesis $0 < \kappa < -\sigma_1$ is not satisfied. If $\kappa \leq 0$, the matrix norm of $A(x)$ cannot be bounded. When$\kappa \geq -\sigma_1$, the norm converges to to a finite value in each Oseledets component destroying the projection onto the stable component.

The proof shows that The constants h_max and h_min allow for local deviations from the long term behavior of the dynamical system , governed by the Lyapunov exponents. If the dynamics are reversible, the least stable Oseledets space becomes the most stable in reverse time. To reverse time we only need to reverse the ordering of the data.

Weighted time delay coordinates not only reconstruct the topology of the but they can also regularize the dynamics.

The projection onto the most stable manifold will often achieve a significant dimensional reduction.

References

Time-Scale Separation from Diffusion-Mapped Delay Coordinates T. Berry, J. R. Cressman, Z. Greguric-Ferencek, and T. Sauer. SIAM Journal on Applied Dynamical Systems, 2013, Vol. 12, No. 2 : pp. 618-649

Strogatz, Steven H. (Steven Henry), (2015). Nonlinear Dynamics and Chaos : with Applications to Physics, Biology, Chemistry, and Engineering. Boulder, CO :Westview Press, a member of the Perseus Books Group